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NEW ALGORITHMS FOR H_∞-NORM APPROXIMATION AND APPLICATIONS

BY

SAMIR HASAN HUSAIN AL-AMER

A Dissertation Presented to the

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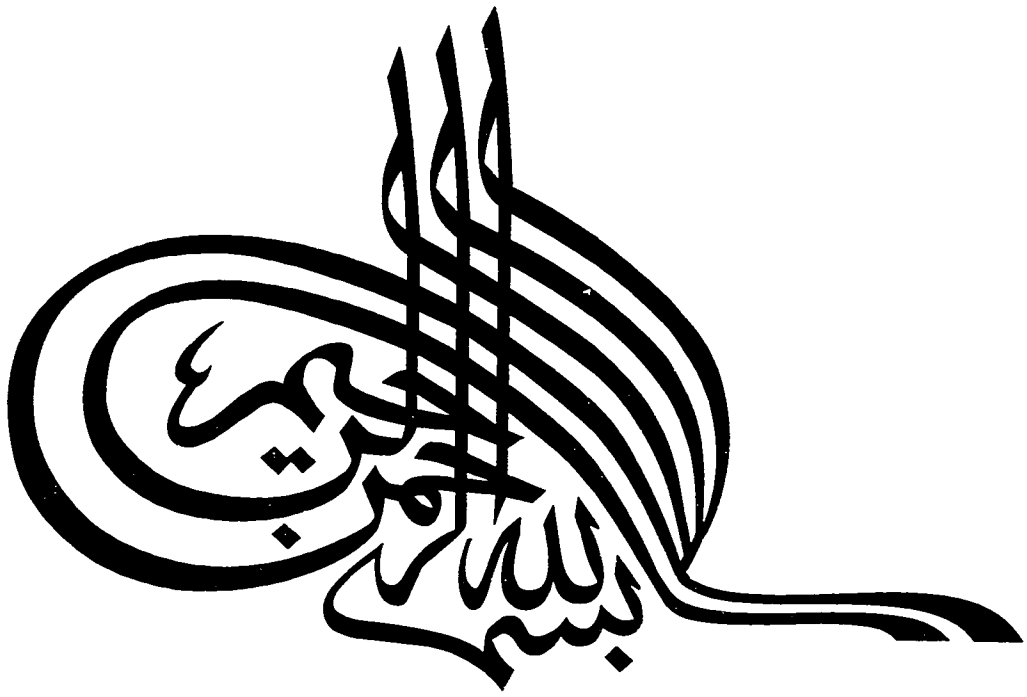
بسم الله الرحمن الرحيم

مهداه الى

أبي وأمي وزوجتي

و أبنائي فاطمة وعبد الله و ريم و بيان

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Nomenclature

$\mathfrak{R}$	Field of real numbers
$\mathfrak{C}$	Field of complex numbers
A^*	Complex conjugate transpose of the matrix A
$*$	Convolution operator
$\mathfrak{R}^{m\times n}$	$m \times n$ matrix with real coefficients
$\mathfrak{C}^{m\times n}$	$m \times n$ matrix with complex coefficients
$\mathfrak{R}_{m,n}(z)$	$\{G(z) = N(z)/D(z)$ where N and D are polynomials of order m and $n\}$
$\overline{z}$	Complex conjugate of z
$\rho(A)$	Spectral Radius of A
$\sigma(A)$	Singular value of A
$\overline{\sigma}(A)$	Maximum singular value of A
$Tr(A)$	Trace of A
$\|G\|_\infty$	$\sup_{\theta\in[0,2\pi]} \overline{\sigma}\left(G(e^{j\omega})\right)$
$\|G\|_2$	$\sqrt{\int_0^{2\pi} Tr[G^*(e^{j\omega})G(e^{j\omega})]\, d\omega}$
$\|G\|_H$	Hankel Norm of G (Definition 3)
H_∞	Space of functions of a complex variable that are analytic in unit disk and bounded on unit circle
RH_∞	Rational transfer functions that belong to H_∞

L_∞	Space of functions of a complex variable that are bounded on the unit circle
L_2	Square integrable transfer functions
H_2	Square integrable stable transfer functions
ℓ_p	Set of all real valued sequences $x(k) \in \Re$ such that $\|x\|_p < \infty$
1-D	One Dimensional
2-D	Two Dimensional
ARMAX	Autoregressive Moving Average with Exogenous Signal
ARX	Autoregressive with Exogenous Signal
BSA	Best Simultaneous Approximation
FFT	Fast Fourier Transform
FIR	Finite Impulse Response
IDFT	Inverse Discrete Fourier Transform
i.i.d	Independent and identically distributed
IIR	Infinite Impulse Response
LCF	Left Coprime Factor
LFT	Linear Fractional Transformation
LMI	Linear Matrix Inequality
LS	Least Squares
LSI	Linear Shift Invariant
LTI	Linear Time Invariant
MIMO	Multiple Input-Multiple Output

MFD Matrix Fraction Description

OE Output Error

RCF Right Coprime Factor

SAP Simultaneous Approximation Problem

SISO Single Input-Single Output

خلاصة الرسالة

اسم الطالب الكامل: سمـــــير حســـــــن حســـــين العامر

عنوان الدراسة: خوارزميات جديده للتقريب باستخدام مقياس أصغر قيمه عظمى و تطبيقاتها

التخصص: هندســـــــــة نظـــــم

تاريخ الدراسة: رمضان ١٤١٩ هـــ

يعتبر التقريب الأمثل باستخدام مقياس أصغر قيمة عظمى من أهم طرق التقريب في مجالات التحكم و معالجة الأشارات. في هذه الدراسة تم اقتراح عدد من الخوارزميات لحل هذا النوع من التقريب كما تم اختيار خمس تطبيقات مهمة في مجالات التحكم و معالجة الأشـــارات لتطبيق هذه الخوارزميات المقترحة و اقتراح تعديلات لحل مسائل خاصة في هذه التطبيقات كما تم استخدامها لحل العديد مـــن المســائل المعروفة في هذا الحقل. الطرق المقترحة هي خوارزميات مكررة و يمكن تقسيمها الى مجموعتين: طرق في المجال الزمني و طـــرق في المجـــال الترددي. كل خطوة في الخوارزميات المقترحة هي عبارة عن حل مسألة تقريب يكون حلها ســـهلاً مقارنـــةً بالمســـألة الأصليـــة. هنـــاك خوارزميات تعتمد على التقريب باستخدام مقياس أقل المربعات و أخرى تعتمد على مقياس خطأ المخرجات. هناك خوارزميات خاصـــة للتقريب الموزون و استخدام التنعيم و استخدام تحويل فوريير السريع و التقريب المقيد و تقريب النظم غير المستقرة و النظم متعددة المتغيرات والنظم ثنائية الأبعاد.

تم استخدام الخوارزميات المقترحة في تصميم المرشحات باستخدام مقياس أقل قيمة عظمى كما تم اقتراح طرق لحل التقريب في حالة التركيز على مقدار الخطأ فقط و في حالات المرشحات ذات الطور الخطي و المرشحات ذات الشكل العام. من التطبيقات المهمة تبسيط المتحكمات و يمكن أيجاد المتحكمات المبسطة عن طريق صياغة المسألة كمسألة تقريب موزون و التي يمكن حلها بأستخدام الخوارزميات المقترحة هنـــا. في هذا البحث تم صياغة تبسيط المتحكمات للنظم الملتبسة كمسألة تقريب موزون.من التطبيقات التي تم دراستها تعين النظم المشكوك بهـــا حيث يتطلب الحصول على النموذج الرياضي للنظام بالأضافة الى حد أقصى لأسوء خطأ ممكن على قياس اصغر قيمة عظمى. تم اقـــتراح خوارزم جديد للتعين لحل هذه المسألة كما تم ايجاد حد أقصى لأسوء خطأ ممكن. تم أيضاً دراسة التقريب المتزامن حيث يتطلب الحصول على دالة واحدة لتقريب مجموعة من الدوال. و قد اقتراح طريقة جديدة لحل هذه المشكلة بصياغتها كمسألة تقريب على قياس اصغر قيمـــة عظمى و تم اقتراح عدد من الخوارزميات لحل المشكلة العامة و حالات خاصة. و أخيرا تم دراسة التقريب للأنظمة ثنائية الأبعاد بأســـتخدام قياس اصغر قيمة عظمى وتم اقتراح خوارزم جديد لحل هذه المسألة.

درجة الدكتوراه في الفلسفة

جامعة الملك فهد للبترول و المعادن

الظهران , المملكة العربية السعودية

رمضان ١٤١٩

DISSERTATION ABSTRACT

FULL NAME OF STUDENT:	Samir Hasan Husain Al-Amer
TITLE OF STUDY:	New Algorithms for H_∞-Norm Approximation and Applications.
MAJOR FIELD:	Systems Engineering
DATE OF DEGREE:	December 1998

In this dissertation, new algorithms to solve the H_∞- norm approximation problem are proposed. These algorithms are iterative and can be classified as time-domain or frequency-domain schemes. The major step in each iteration involves the solution of an identification problem. Methods based on Output Error and Least Squares are proposed, as are extensions based on the use of smoothing, Fast Fourier Transform, constrained approximation, approximation of unstable systems, multivariable systems and two-dimensional systems. The proposed algorithms are applied to solve several well-known problems in control and signal processing. Five applications are considered and in each application several examples and case studies are analyzed. One important application is the minimax IIR filter design. Those algorithms have been used to solve magnitude-only approximation, linear phase IIR filters and general shape filters. The controller reduction problem can be converted to a weighted approximation problem that can be solved by the proposed algorithms. A new scheme for controller reduction of uncertain systems has also been proposed. Furthermore, a new algorithm for identification in H_∞ has been presented. In this approach the nominal model is obtained by solving a weighted H_∞-norm approximation problem and then a bound on the worst case error is obtained. The H_∞-simultaneous approximation problem can be solved by posing the problem as a weighted H_∞-norm approximation problem. Algorithms to solve the general and special cases of the simultaneous approximation problem are also studied. Approximation of two-dimensional systems has been addressed and a new H_∞-norm approximation algorithm for 2-D systems is given.

DOCTOR OF PHILOSOPHY DEGREE
KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
Dhahran, Saudi Arabia
December 1998

Chapter 1

INTRODUCTION

1.1 Introduction

Construction of mathematical models is an important step in the analysis and design of systems. There are two major criteria in coming up with these models: accuracy and simplicity. In general they are conflicting and a trade off is needed. The simplicity of a model depends on the model structure and on the number of parameters used. A linear model is, in general, much simpler than a nonlinear model having the same number of parameters or even less. In two-dimensional systems, a separable system is much simpler than a non-separable one. Among systems of the same type a small number of parameters means simplicity and for this reason model reduction is often used to mean approximation.

Modeling of physical systems often leads to very high order or even infinite dimensional models. Performing analysis, controller design and simulation of high order systems require a huge volume of computations. Model reduction leads to considerable reduction in the required computation. The availability of low order models may result in simpler controller structures.

Approximation or model reduction is an active area of research and a large amount huge literature is available (See, for example, the survey papers [1-3]). Model reduction algorithms are classified as frequency-domain or time-domain techniques. They are also classified as optimal or non-optimal techniques. Non-optimal techniques are based on intuitive ideas and the key features of the original model are retained in the reduced order model. It is well known that the dominant poles - poles closer to the $j\omega$ axis in continuous time systems contribute much to the system response. A classical technique, the dominant-pole of Davison, tries to keep the dominant poles and their corresponding eigenvectors in the reduced order model and to cancel the others [4]. The impulse response of an LTI system determines its response characteristics. Padé techniques try to match the first few terms of the impulse response of both the original and the reduced order models. An important technique that has gained popularity in the last two decades is the balanced truncation algorithm [5]. Balanced truncation removes the states that are weakly controllable and observable. Such states have little effect on the input-output behavior of the system.

In optimal techniques, an optimality criterion is used to select an approximate model that is best with respect to that criterion. A meaningful approximation technique is one that minimizes the "approximation error". There are many ways of defining the approximation error. Commonly used approximation criteria include least squares and ∞-norm. The selection of the approximation error is application dependent and in general a good approximation with respect to one criterion may not be good with respect to another. Wilson [6] minimizes the integral squared impulse response error between the full and the reduced order model. El-Attar and Vidyasagar [7] minimized the ℓ_1-norm of the error in impulse response. Frequency domain techniques were considered by Reddy [8], Luss [9] and many others. Other authors tried com-

binations of time-domain and frequency-domain criteria [7, 10]. Another popular technique is the optimal Hankel norm model reduction [11]. The H_∞-norm approximation technique is one of the most important techniques used in the current literature. It has strong physical interpretations and many meaningful problems can be formulated as H_∞-norm approximation problems.

1.2 H_∞-Norm Model Reduction Problem

To solve any approximation problem one needs to select the family of approximating functions and the approximation measure. Next, the existence and uniqueness of a solution needs to be investigated. A third step is to obtain the characteristics or the special properties of the solutions and finally the computational algorithm is developed.

As a well known problem in literature, the H_∞-norm model reduction problem is also known as the minimax approximation and complex Chebyshev rational approximation. It is concerned with finding a transfer function such that the maximum amplitude of the approximation error is minimized. The mathematical definition of the problem will be given in Chapter 2. The solution to this problem is known to exist but it is not necessarily unique [12]. The characterization of the optimal solution is non-trivial. Necessary conditions and sufficient conditions for optimality have been obtained in [12-15].

If the approximating function is assumed to be rational, then the approximation problem is a nonlinear approximation problem and the computation of the optimal H_∞-norm reduced order model is still an open problem. Numerical optimization algorithms to solve the problem often exhibit undesirable properties such as: high computation time, sensitivity to starting points

and lack of robustness. If the approximating function is restricted to polynomial functions, the problem is reduced to the well-studied polynomial approximation that is much easier to solve.

Suboptimal solutions to the H_∞-norm approximation problem can be obtained using balanced truncation [5] and optimal Hankel approximation [11]. The balanced model reduction and Hankel model reduction are not directly related to H_∞-norm model reduction, but ∞-norm bounds on the deviation error are available. Such error bounds give indications about the accuracy lost due to the approximation.

State-space techniques such as Hankel approximation and balanced truncation involve a larger number of parameters compared to transfer functions. For an n^{th} order SISO system, there are $2n+1$ parameters in the transfer function compared to n^2+2n+1 in the equivalent state-space model. The algorithms proposed in this work are transfer function-based algorithms. More details on the existing algorithms are given in Chapter 2.

1.3 Applications of H_∞-Norm Model Reduction

Model reduction is often used to make significant reduction in the computational burden needed in the analysis and the simulation of physical systems. In addition, many problems in the control and the signal processing literature can be formulated as H_∞-norm approximation problems. Efficient solution of such applications is possible with an efficient H_∞-norm approximation algorithm. In this dissertation, several applications of the H_∞-norm model reduction are considered, some of which are given below.

1.3.1 Minimax Filter Design

The design of digital filters is a well-known problem and many algorithms have been used to solve it. The minimax filter is an important class of filters. Given ideal filter characteristics and a desired filter order, the minimax filter design problem can be stated as: find a filter of the specified order such that the maximum magnitude of the error between the ideal filter and the designed filter is minimized. This can be easily formulated as a weighted H_∞-norm approximation problem. In Chapter 4, several filter design problems are formulated as weighted H_∞-norm approximation problems. Examples of such filters are discussed below.

1. Linear Phase IIR Filters: Linear phase filters are desirable in many applications. FIR linear phase filters can be easily designed but their orders can be very high for complex design problems. The design of linear phase IIR filters can be obtained as the H_∞-norm approximation of the FIR filters.

2. General Shape Filters: Standard filter design techniques are restricted to piecewise constant frequency characteristics. The design of general shape filters can be formulated as the weighted H_∞-norm approximation of the ideal specifications.

1.3.2 Controller Reduction

It is often desirable to use low order controllers because they are easier to implement and are more reliable. Many controller design algorithms produce controllers of orders similar to that of the plant order or larger.

An important technique used to obtain low order controllers is known as controller reduction. A high order controller is initially designed to satisfy the specifications then a lower order

approximation to the original controller is obtained. Simple applications of model reduction techniques may result in controllers that are not guaranteed to stabilize the closed loop system. Controller reduction techniques which guarantee robust stability and/or performance can be formulated as weighted H_∞-norm approximation problems. This problem will be further investigated in Chapter 5.

1.3.3 Identification in H_∞

When identifying a system, one is not expected to get a perfect match of the true and the identified model. The difference between the two can be expressed as an additive perturbation and it is desirable to get the smallest possible mismatch. The H_∞-identification algorithms use frequency response measurements together with a priori information on the system to be identified to come up with a nominal model and a bound on the worst case additive perturbation. The availability of the nominal model and an upper bound on the error is essential for the design of robust controllers. In order to come up with the bound, algorithms that guarantee a bound on the error are used. With this restriction, the obtained nominal model may not be a good approximation of the system. In Chapter 6, the problem is formulated and a new algorithm is proposed. The obtained nominal model is the best approximation, in ∞-norm sense, to the experimental frequency response measurements.

1.3.4 H_∞-Norm Simultaneous Approximation

The H_∞-norm simultaneous approximation problem is to find a single system that best approximates, in ∞-norm sense, a given family of systems. Many problems in control systems can be formulated as H_∞-norm simultaneous approximation problems. Examples of possible

applications are given below.

1. Multi-objective Controller Reduction: A very high order controller can be designed to achieve stability and performance specifications. The problem of designing reduced order controllers that achieve both robustness and performance or different specifications can be formulated as a simultaneous approximation problem.

2. Linearization of Nonlinear Systems: A nonlinear system can be linearized about a specified operating condition. Suppose it is desired to obtain a linear model that approximates a nonlinear system at M different operating points. This problem can be formulated as a simultaneous approximation problem. The nonlinear system is linearized at the M operating conditions, which results in M linear models. A single model is then obtained to simultaneously approximate the M models.

3. Nominal Model Selection: Consider the problem of obtaining a nominal model for a system with parametric uncertainty. One can formulate the problem as simultaneous approximation as follows. First, M possible combinations of the parameters are obtained to achieve M different realizations. The nominal model is then obtained to simultaneously approximate the M models.

The precise definition and the new solution algorithms are given in Chapter 7.

1.3.5 Two-Dimensional Approximation

Approximation of two-dimensional systems has many applications in 2-D filter design and image compression. The problem is more complicated than the 1-D approximation. This is due to the nature of the 2-D systems and to the large volume of data to be processed. Given the

specifications of a 2-D ideal filter, the filter design problem can be formulated as the H_∞-norm approximation of a 2-D system. This will be discussed in Chapter 8.

1.4 Contribution of the Dissertation

In this dissertation, several algorithms to solve the H_∞-norm approximation problem are proposed. The general approach used in all algorithms is to solve iteratively weighted approximation problems. The algorithms are grouped into two classes: time-domain and frequency-domain algorithms. The performance of the proposed algorithms is also investigated. The algorithms and their extensions are used to solve several well-known problems in control theory and signal processing. Algorithms to solve the IIR Filter design problem with or without phase constraints are proposed. The algorithms can be used to obtain the IIR filter directly from the ideal frequency specifications. The frequency domain algorithm is used to solve the controller reduction for uncertain systems. Based on the frequency domain algorithms, new algorithms to solve the H_∞-norm simultaneous approximation problems are proposed. A new algorithm for identification in H_∞ is proposed. The nominal model is obtained by solving an H_∞-norm approximation problem. Finally, an algorithm for approximating 2-D systems is proposed. Several papers based on this research have been published [16-19] and several others are under preparation.

1.5 Dissertation Outline

The dissertation is divided into nine chapters. A brief summary of the remaining chapters is given below.

Chapter 2: The H_∞-norm model reduction problem is investigated and existing methods to

solve it are discussed.

Chapter 3: This is the main chapter of the dissertation. The new approximation algorithms are presented and their properties are investigated. They are grouped as time and frequency-domain algorithms. Implementation issues and special cases are also considered.

Chapter 4: The minimax IIR filter design problem is formulated as a weighted H_∞-norm approximation problem that can be directly solved by the algorithms of Chapter 3. Modified algorithms are derived for special filter design problems.

Chapter 5: The approximation algorithms are used to solve controller reduction problems. In this chapter, the controller reduction problem is formulated as a weighted model reduction problem and then solved using the algorithms of Chapter 3.

Chapter 6: Identification in H_∞ is concerned with obtaining a nominal model representing the identified system as well as a bound on additive approximation error. The nominal model is obtained from the noisy measurement by solving an H_∞-norm approximation problem. The experimental data together with a priori information about the identified model are used to come up with the error bound.

Chapter 7: Simultaneous approximation is concerned with finding a single system that approximates a family of systems. This problem is solved by formulating the problem as a weighted model reduction problem.

Chapter 8: The frequency-domain algorithms of Chapter 3 are extended to approximate the 2-D systems.

Chapter 9: Conclusions and recommendations for future research are reported in this chapter.

Chapter 2

PROBLEM STATEMENT AND EXISTING ALGORITHMS

2.1 Introduction

In this chapter, the H_∞-norm approximation problem is defined and existing solution algorithms are discussed. First, general material needed for the remaining chapters is given in Section 2.2, then the H_∞-norm approximation problem is defined in Section 2.3 and existing solution algorithms are discussed in Section 2.4.

2.2 Preliminaries

In this section, preliminary materials are given. The objective is to set the notation and to give basic materials that are used throughout the dissertation.

2.2.1 Linear Time-Invariant Systems

Consider a discrete-time linear shift invariant system with q-inputs and p-outputs. It can be described by

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\,\mathbf{x}_k + \mathbf{B}\,\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}\,\mathbf{x}_k + \mathbf{D}\,\mathbf{u}_k. \end{aligned} \tag{2.1}$$

We will use $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$ to represent the above system. The corresponding transfer function is given by

$$G(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$$

Two important norms used throughout the dissertation are defined below.

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} Tr[\mathbf{G}^*(e^{j\omega})\mathbf{G}(e^{j\omega})]\; d\omega}$$

$$\|G\|_\infty = \sup_{\omega\in[0,2\pi]} \overline{\sigma}\left(\mathbf{G}(e^{j\omega})\right). \tag{2.2}$$

The first one is called the 2-norm and the second one is the ∞-norm.

2.2.2 Controllability and Observability Gramians

Definition 1 *Consider a stable transfer function $G(z)$. Let* $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$ *be a state-space realization of $G(z)$. The Controllability Gramian $\mathbf{P}$ and the Observability Gramian $\mathbf{Q}$ are defined*

as:

$$\mathbf{P} = \sum_{k=0}^{\infty} \mathbf{A}^k \mathbf{B}\mathbf{B}^T \left(\mathbf{A}^T\right)^k,$$
$$\mathbf{Q} = \sum_{k=0}^{\infty} \left(\mathbf{A}^T\right)^k \mathbf{C}^T\mathbf{C}\mathbf{A}^k.$$

It is clear that, $\mathbf{P}$ and $\mathbf{Q}$ satisfy the Lyapunov equations

$$\mathbf{A}\mathbf{P}\mathbf{A}^T - \mathbf{P} = -\mathbf{B}\,\mathbf{B}^T,$$
$$\mathbf{A}^T\mathbf{Q}\mathbf{A} - \mathbf{Q} = -\mathbf{C}^T\mathbf{C}.$$

Lemma 1 *[20] A state-space realization is controllable if and only if* $\mathbf{P}$ *is positive definite. It is observable if and only if* $\mathbf{Q}$ *is positive definite.*

2.2.3 Hankel Singular Values

Consider a discrete-time linear system with the transfer function

$$G(z) = \sum_{k=0}^{\infty} g_k \, z^{-k}$$

The corresponding Hankel matrix is defined as the following infinite dimensional matrix:

$$\mathbf{H}_G = \begin{bmatrix} g_1 & g_2 & g_3 & \cdot & \cdot & \cdot \\ g_2 & g_3 & \cdot & \cdot & \cdot & \cdot \\ g_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

It is well known [20] that the minimal order of a linear system is equal to the rank of the associated Hankel matrix. The singular values of $\mathbf{H}_G$ are called the Hankel singular values (HSV) of the system $G(z)$. The following lemma defines the HSV in terms of the controllability and observability Gramians.

Definition 2 *Consider an n^{th} order transfer function $G(z)$ with a minimal realization* $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$. *Let $\mathbf{P}$ and $\mathbf{Q}$ be the controllability and observability Gramians respectively. The HSV are defined as*

$$\sigma_i(G) = \lambda_i^{\frac{1}{2}}(\mathbf{PQ}). \tag{2.3}$$

The HSV are input-output invariant and are usually ordered so that $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant 0$.

Definition 3 *The Hankel norm of a transfer function $G(z)$, denoted by $\|G\|_H$, is defined as the largest HSV of $G(z)$, i.e.,*

$$\|G\|_H = \sigma_1.$$

The Hankel norm is always less than or equal to the ∞-norm of the system:

$$\|G\|_H \leq \|G\|_\infty.$$

2.2.4 Linear Matrix Inequalities

A linear matrix inequality (LMI) is given as

$$F(x) = F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_N F_N \leq 0$$

where x_i are real numbers and F_i are symmetric matrices. Efficient algorithms to solve LMI problems are now available and consequently many problems in control theory are formulated and solved by LMI algorithms (See for example, [21], [22]). In the following we formulate one such problem as an LMI problem.

Given a stable system $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, $\|G\|_\infty < \gamma$ if and only if there exists a symmetric positive definite matrix P such that

$$\begin{bmatrix} PA + A^T P & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0.$$

This is called the Bounded Real Lemma [21].

2.3 Optimal H_∞-Norm Model Reduction Problem

The optimal H_∞-norm model reduction problem is known in mathematical literature as the complex Chebyshev rational approximation.

Definition 4 *Given an n^{th} degree $p \times q$ stable transfer function $G_n(z)$, find a rational transfer function $G_r(z)$ having McMillan degree r that minimizes $\|G_n - G_r\|_\infty$. A generalization of this problem is the weighted H_∞-norm model reduction problem where G_r is obtained such that $\|W_i(G_n - G_r)W_o\|_\infty$ is minimum.*

The weighting functions $W_i(z)$ and $W_o(z)$ are used to emphasize errors in certain ranges of frequencies. In the above problem, the reduced order model $G_r(z)$ is assumed to be a rational function of the complex variable 'z'. For the SISO case, the approximate model is assumed to have the following form

$$G_r(z) = \frac{n_r(z)}{d_r(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_r z^{-r}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_r z^{-r}} .$$

In 1931, Walsh [12] established that there exists a rational solution $G_r(z)$ that best approximates a complex valued function $G(z)$ on any arbitrary compact subset of the complex plane $\mathfrak{C}$ with no isolated points. It is well known that the solution is not always unique. In fact non-uniqueness is quite common in rational approximation. Several authors investigated the non-uniqueness phenomena. Gutknecht and Trefethen [23] showed that the best complex rational Chebyshev approximation on the unit disk need not be unique and the number of best approximations can be arbitrarily large.

2.3.1 Geometric Interpretation of the H_∞ Approximation

The H_∞ -norm approximation problem can be formulated as follows: Given a rational transfer function $G_n(z)$, find $G_r(z)$ of the desired order so that the error function $E(z) = G_n(z) - G_r(z)$ is contained in a disk of minimal radius centered at the origin. Using the maximum modulus principle, the maximum occurs on the boundary and therefore we restrict our attention to the unit circle (i.e., $E(e^{j\omega}) = G_n(e^{j\omega}) - G_r(e^{j\omega})$). It was observed experimentally that the error curve for the optimal approximation $G_r^*(z)$ often has the shape of a circle. For low-order approximation, the error function may not have the shape of a circle but as the order increases, it will get closer to the shape of a circle. It was proved in [24] that if the error curve is a perfect circle with a winding number greater than or equal to $n+m+1$, then $G_r(z)$ is a best rational approximation of $G_n(z)$ over all $G_r(z) \in \mathfrak{R}_{n,m}(z)$.

However, as one might expect, a perfect circular error can not, in general, be achieved. It has been shown that near circularity of the error implies that $G_r(z)$ is the near-best Chebyshev approximation of $G_n(z)$ [24].

Lemma 2 *[24] Suppose the error curve $G\left(e^{j\omega}\right) - G_r(e^{j\omega})$ has a winding number of at least $2r+1$ about the origin. Then*

$$\min_{\omega} \left|G_n\left(e^{j\omega}\right) - G_r\left(e^{j\omega}\right)\right| \leq \left|G_n\left(e^{j\omega}\right) - G_r^o\left(e^{j\omega}\right)\right| \leq \|G_n - G_r\|_\infty$$

where $G_r^o\left(e^{j\omega}\right)$ is an optimal approximant.

The above lemma can be very useful in bounding the approximation error. Given any approximant $G_r(z)$, one can compute the maximum and minimum values of the approximation

error and these values represent upper and lower bounds on the optimal approximation error. If for a given $G_r(z)$, the upper and lower bounds are equal then the error is perfectly circular and $G_r(z)$ is the optimal approximation.

2.3.2 Characterization of Optimal Approximation

A major step in solving any approximation problem is to understand different properties of the solution and to come up with a characterization of it; A computational algorithm can then be developed based on this characterization. A necessary condition for the optimal solution is given by the local Kolmogorov condition and Ruttan's theorem gives a sufficient optimality condition [14]. Unfortunately it is difficult to convert such conditions into an approximation algorithm.

Kavranoğlu and Bettayeb [25-29], characterized the optimal r^{th} order approximant and proposed an algorithm to compute it. The r^{th} order model reduction is converted to an all pass imbedding problem. The following lemma provides a characterization of the optimal r^{th} order approximation.

Lemma 3 *[25, 28] Given an n^{th} order $m \times m$ transfer function G_n with a minimal realization* $G_n(z) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$, *any r^{th} order G_r satisfying*

$$\|G_n - G_r\|_\infty \leq \gamma$$

is characterized as follows: Compute $\mathbf{B}_0$ *and* $\mathbf{C}_0$ *such that*

$$\mathbf{APA}^T - \mathbf{P} + \mathbf{BB}^T + \mathbf{B}_0\mathbf{B}_0^T = 0 \tag{2.4}$$

$$\mathbf{A}^T\mathbf{Q}\mathbf{A} - \mathbf{Q} + \mathbf{C}^T\mathbf{C} + \mathbf{C}_0^T\mathbf{C}_0 \;=\; \mathbf{0},$$

and

$$\lambda_{\min}(\mathbf{QP}) = \gamma^2 \quad \textit{with multiplicity } n - r \tag{2.5}$$

where $\gamma \geq \gamma_0$ *with* γ_0 *being the minimum achievable error. The* r^{th} *order approximation can be obtained as*

$$G_r(z) = (I \;\; 0)\, H_r(z) \begin{pmatrix} I \\ 0 \end{pmatrix}$$

where $H_r(z)$ *is the* r^{th} *order optimal Hankel approximation of*

$$\left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B} & \mathbf{B}_0 \\ \hline \mathbf{C} & 0 & 0 \\ \mathbf{C}_0 & 0 & 0 \end{array}\right].$$

The above lemma was proved for the continuous-time case in [25] and for the discrete-time case in [28]. The main difficulty in this theorem is finding $\mathbf{B}_0$ and $\mathbf{C}_0$ that correspond to the optimal γ.

2.4 Review of Known Algorithms

The H_∞-norm approximation problem is a very important problem and many solution procedures have been proposed but so far there is no globally accepted algorithm to solve the general version of the problem. In this section, several suggested approaches for solving the above problem are discussed. These include balanced truncation, the Hankel approximation, methods

based on optimal characterization, differential correction, complex curve fitting and iterative reweighted least squares methods.

2.4.1 Balanced Realization and Truncation

The balanced realization first appeared in 1976 in the work of Mullis and Roberts [30]. It was used in the design of filters with good round-off error characteristics. The balanced realization is defined as follows.

Definition 5 *A state-space realization* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is balanced if the controllability and observability Gramians are equal and diagonal*

$$\mathbf{P} = \mathbf{Q} = \sum = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n \end{bmatrix} \tag{2.6}$$

where σ_i *are the HSV of the system* $G(z) = C(zI - A)^{-1}B + D$.

Computing Balanced Realization

Every stable, minimal realization has a balanced realization. There are several algorithms used for obtaining the balanced realization of a given system (see for example [5, 31-34]). A simple algorithm due to Moore [5] is as follows.

***ALGORITHM* 2.1: Balancing Algorithm** [5]

Given a minimal state-space realization $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$,

Step 1: Compute the controllability and observability Gramians $\mathbf{P}$ and $\mathbf{Q}$,

Step 2: Find the Cholesky decomposition $\mathbf{Q} = \mathbf{R}^T\mathbf{R}$ of $\mathbf{Q}$.

Step 3: Diagonalize the symmetric positive definite matrix $\mathbf{RPR}^T$ as

$$\mathbf{RPR}^T = \mathbf{U}\boldsymbol{\Sigma}^2\mathbf{U}^T,$$

where $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ and $\boldsymbol{\Sigma}$ is given by (2.6).

Step 4: Let the transformation matrix $\mathbf{T}$ be given by

$$\mathbf{T} = \boldsymbol{\Sigma}^{-1/2}\mathbf{U}^T\mathbf{R}$$

then, a balanced realization is given by $\left[\begin{array}{c|c} \mathbf{TAT}^{-1} & \mathbf{TB} \\ \hline \mathbf{CT}^{-1} & \mathbf{D} \end{array}\right]$.

Balanced Truncation

The states of a balanced system are equally controllable and observable. In 1981, Moore [5] introduced the balanced truncation algorithm. The basic idea here is to discard the states that are weakly controllable and observable. Such states have little effect on the behavior of the system. This method has become a very popular approximation technique. Balanced model reduction has two main advantages: First the reduced model is guaranteed to be stable. Moreover, upper and lower bounds on the approximation error are available.

Let $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$ be a balanced realization of an n^{th} order system $G_n(z)$. Partition the system matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as follows.

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix}$$

where $\mathbf{A}_{11} \in \Re^{r\times r}$, $\mathbf{B}_1 \in \Re^{r\times p}$, $\mathbf{C}_1 \in \Re^{q\times r}$. A balanced truncation of order r is given by

$$G_r(z) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right].$$

Two important features of the balanced realization are given by the following lemmas.

Lemma 4 *[35] Given an n^{th} order state-space realization* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *of a stable transfer function G_n. Let $G_r(z)$ be the r^{th} order balanced truncation model, then*

$$\sigma_{r+1} \leq \|G_n - G_r\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i \tag{2.7}$$

The left hand side inequality is the well-known Nehari theorem. The right hand side inequality was first proved by Enns [32] for the continuous-time case, and the discrete-time case was proved by Al-Saggaf and Franklin [35, 36].

Lemma 5 *[36] Let G_n be a stable system with a balanced state-space realization*

$$\left[\begin{array}{cc|c} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{B}_1 \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{C}_2 & \mathbf{D} \end{array}\right]$$

with $\mathbf{A}_{11} \in \Re^{r\times r}$, and assuming that $\sigma_r > \sigma_{r+1}$, then the balanced truncated model $\left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{B}_1 \\ \hline \mathbf{C}_1 & \mathbf{D} \end{array}\right]$ *is balanced and asymptotically stable.*

Note that Lemmas 4 and 5 are true for both continuous and discrete-time systems.

Weighted Balanced Truncation

The weighted approximation problem is to find $G_r \in \Re H_\infty$ that minimizes

$$\|W_i(G_n - G_r)W_o\|_\infty. \tag{2.8}$$

A solution to this problem was proposed by Enns [32]. There are two major problems with the weighted balanced truncation algorithm. Except for very special cases, there is no known error bound similar to those given by equation (2.7). Moreover, $G_r(z)$ is not guaranteed to be stable. Alternative formulations provide error bounds for some classes of weighting functions (See, for example, [35], [37], [38]). Unfortunately, for most problems the error bounds are not reasonably tight. Algorithms that guarantee stability of the reduced order model have been recently proposed [39]. In the following, we present a summary of the algorithm reported in [38].

***ALGORITHM* 2.2: Weighted Balanced Truncation** [38]

Given $W_i(s)$, $W_o(s)$, $G_n(s)$ all in $\Re H_\infty$ and having the following state-space realizations

$$\mathbf{W}_o = \left[\begin{array}{c|c} \mathbf{A}_o & \mathbf{B}_o \\ \hline \mathbf{C}_o & \mathbf{D}_o \end{array}\right], \mathbf{W}_i = \left[\begin{array}{c|c} \mathbf{A}_i & \mathbf{B}_i \\ \hline \mathbf{C}_i & \mathbf{D}_i \end{array}\right], G_n = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$$

Step 1: Define

$$\hat{\mathbf{A}}_i = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{C}_i \\ \mathbf{0} & \mathbf{A}_i \end{bmatrix}, \quad \hat{\mathbf{A}}_o = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B}_o\mathbf{C} & A_o \end{bmatrix} \quad \hat{\mathbf{B}}_i = \begin{bmatrix} \mathbf{B}\mathbf{D}_i \\ \mathbf{B}_i \end{bmatrix}, \quad \hat{\mathbf{C}}_o = \begin{bmatrix} \mathbf{D}_o\mathbf{C} & \mathbf{C}_o \end{bmatrix}$$

Step 2: Let $\hat{\mathbf{P}}, \hat{\mathbf{Q}}$ having the form

$$\hat{\mathbf{P}} = \begin{bmatrix} \mathbf{P} & \mathbf{P}_{12} \\ \mathbf{P}_{12}^T & \mathbf{P}_{22} \end{bmatrix}, \quad \hat{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q} & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}^T & \mathbf{Q}_{22} \end{bmatrix}$$

be the solution of the Lyapunov equations

$$\hat{\mathbf{A}}_i\hat{\mathbf{P}} + \hat{\mathbf{P}}\hat{\mathbf{A}}_i^T = -\hat{\mathbf{B}}_i\hat{\mathbf{B}}_i^T$$

$$\hat{\mathbf{A}}_o^T\hat{\mathbf{Q}} + \hat{\mathbf{Q}}\hat{\mathbf{A}}_o = -\hat{\mathbf{C}}_o^T\hat{\mathbf{C}}_o$$

Step 3: Find a nonsingular matrix $\mathbf{T}$ such that

$$\mathbf{T}\mathbf{P}\mathbf{T}^T = (\mathbf{T}^{-1})^T\mathbf{Q}\mathbf{T}^{-1} = \mathbf{\Sigma} = \text{diag}(\mathbf{\Sigma}_1, \mathbf{\Sigma}_2)$$

Step 4: The weighted balanced transformation is given by

$$G_{wb} = \left[\begin{array}{c|c} \mathbf{TAT^{-1}} & \mathbf{TB} \\ \hline \mathbf{CT^{-1}} & \mathbf{D} \end{array}\right]$$

Step 5: Now partition G_{wb} as follows

$$G_{wb} = \left[\begin{array}{cc|c} \mathbf{A_{11}} & \mathbf{A_{12}} & \mathbf{B_1} \\ \mathbf{A_{21}} & \mathbf{A_{22}} & \mathbf{B_2} \\ \hline \mathbf{C_1} & \mathbf{C_2} & \mathbf{D} \end{array}\right]$$

where

$$\mathbf{A_{11}} \in \Re^{r\times r}, \mathbf{A_{22}} \in \Re^{n-r\times n-r}$$

The r^{th} order weighted balanced truncation is given by

$$G_r = \left[\begin{array}{c|c} \mathbf{A_{11}} & \mathbf{B_1} \\ \hline \mathbf{C_1} & \mathbf{D} \end{array}\right].$$

2.4.2 H_∞ Model Reduction Based on the Optimal Hankel Norm Approximation

The Hankel operator of a linear system is the prediction operator that maps past inputs to future outputs assuming the future input to be zero. Some of the earlier results on the Hankel norm approximation can be found in [40-44].

In the following, we outline the H_∞-norm approximation based on the solution of the optimal Hankel approximation problem. Despite the fact that the Hankel approximation is not directly

related to the ∞-norm, the algorithm discussed here is commonly used in literature to obtain a suboptimal H_∞-norm approximation.

***ALGORITHM* 2.3: Optimal Hankel Norm Approximation**

Given an n^{th} order model $G_n(s) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$ and the desired order r,

Step 1: Compute the controllability and observability Gramians $\mathbf{P}$ and $\mathbf{Q}$.

Step 2: Select orthogonal vectors $\mathbf{x}_i \in \Re^n$ such that

$$\mathbf{PQ}\mathbf{x}_i = \sigma_i^2 \mathbf{x}_i$$

$$\mathbf{x}_i^T \mathbf{Q} \mathbf{x}_i = \sigma_i^2$$

Step 3: Compute the Schmidt pairs w_{r+1} and v_{r+1} associated with σ_{r+1} using

$$w_i(t) = \sigma_i^{-1} \mathbf{C} e^{At} x_i$$

$$v_i(t) = \sigma_i^{-2} \mathbf{B}^T e^{\mathbf{A}^T t} \mathbf{Q} x_i$$

Step 4: Compute $\hat{x}$ from

$$(G + \hat{x})(\lambda) = \sigma_{r+1} \frac{w_{r+1}(\lambda)}{v_{r+1}(\lambda)}$$

Step 5: The optimal Hankel norm reduced order model G_r is the causal part of $\hat{x}$.

Glover [11], gave state-space construction of all optimal Hankel norm approximations. Extension to the frequency weighted case was done by Lantham and Anderson [45] for the scalar case and by Hung and Glover [37] for the matrix case.

Let $G_n(z) \in RL_\infty$ be an n^{th}order transfer function and let $\hat{G}$ be its optimal r^{th} order Hankel

norm approximation, then it is clear that

$$\|G_n - \hat{G}\|_H = \sigma_{r+1}.$$

Recall that the D term has no effect on the HSV and as a result it has no effect on the Hankel norm and therefore

$$\|G_n - \hat{G} - D\|_H = \sigma_{r+1}$$

for any constant matrix D. The ∞-norm, on the other hand, is affected by the D term. Glover [11] showed that there exists a constant matrix $\hat{D}$ such that

$$\sigma_{r+1} \leq \|G_n - \hat{G} - \hat{D}\|_\infty \leq \sum_{i=r+1}^{n} \sigma_i. \tag{2.9}$$

The right hand side of (2.9) can be further reduced if some of the HSV are repeated. To obtain a suboptimal H_∞-norm approximation, the following algorithm can be used.

***ALGORITHM* 2.4: H_∞ Approximation via Optimal Hankel Norm Approximation**

Given $G_n = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and the desired order r.

Step 1: Find $\hat{G}$, the optimal Hankel norm approximation of G_n,

Step 2: Find a constant matrix $\hat{D}$ which minimizes

$$\|(G_n - \hat{G}) - \hat{D}\|_\infty,$$

The suboptimal r^{th} order L_∞ approximation of G_n is given by

$$G_r = \hat{G} + \hat{D}$$

Using the above algorithm, the error bound is given by (2.9) which is tighter than that of the balanced truncation. This is the main motivation behind using this approximation technique. The zeroth order approximation algorithm in [46] can be used to find the constant matrix $\hat{D}$. An alternative approach is to convert the problem into an LMI problem which can be solved efficiently by the LMI toolbox of MATLAB.

2.4.3 Algorithms Based on Characterization of Optimal H_∞-Norm Approximation

In Section 2.3.2, a state-space characterization of the optimal H_∞-norm approximation was given. The main difficulty is to find $\mathbf{B_0}$ and $\mathbf{C_0}$ that correspond to the minimum approximation error. In the following, we outline the suboptimal approximation procedure of [28]. Let the solution of the Lyapunov equation (2.4) be decomposed as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{pmatrix} \text{ with } \mathbf{Q}_1,\ \mathbf{P}_1 \in \Re^{r\times r} \text{and } \mathbf{Q}_2,\ \mathbf{P}_2 \in \Re^{n-r\times n-r}$$

In the suboptimal reduced order model, $\mathbf{B_0}$ and $\mathbf{C_0}$ are obtained so that

$$\mathbf{P}_2\,\mathbf{Q}_2 = \gamma^2 \mathbf{I}_{n-r}. \tag{2.10}$$

It is clear that (2.10) is more restrictive than (2.5).

***ALGORITHM* 2.5: State-Space H_∞-Norm Model Reduction[28]**

Given an n^{th} order $G(z) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right]$, and a positive number γ,

Step 1: Find $\mathbf{B_0}$ and $\mathbf{C_0}$ such that

$$\mathbf{P\,Q} = \gamma^2 \mathbf{I}_{n-r}, \tag{2.11}$$

such that

$$\begin{aligned} \mathbf{APA}^T - \mathbf{P} + \mathbf{BB}^T + \mathbf{B_0B_0}^T &= \mathbf{0} \\ \mathbf{A}^T\mathbf{QA} - \mathbf{Q} + \mathbf{C}^T\mathbf{C} + \mathbf{C_0}^T\mathbf{C_0} &= \mathbf{0}, \end{aligned} \tag{2.12}$$

Step 2: Find $H_r(z)$, the r^{th} order Hankel approximation of

$$\left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B} & \mathbf{B_0} \\ \hline \mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{C_0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

Step 3: $G_r(z) = (\mathbf{I} \quad \mathbf{0}) H_r(z) \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix}$.

The problem is considerably simplified. For details of computation involved consult [28].

An alternative approach was introduced in [29] which is based on linear matrix inequality

formulation. The problem of obtaining the optimal H_∞-norm approximation can be stated as:

$$\min_{\alpha, P=P^T, R=R^T, M=M^T>0} \quad \alpha$$

subject to

$$\begin{aligned} AP + PA^T + BB^T &\leq 0 \\ \begin{bmatrix} AR + RA^T & RC^T \\ CR & -\alpha I \end{bmatrix} &\leq 0 \\ R &\leq P \\ P - R &\leq M \text{ with } rank(M) = r \end{aligned}$$

The values of Q and R are related as follows

$$Q = \gamma^2 R^{-1}.$$

The last constraint is nonconvex and because of this the above problem is not an LMI. A suboptimal approximation can be obtained by assuming that

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} \text{ with } M_1 \in R^{r \times r}.$$

With this assumption the above problem turns out to be an LMI. An iterative approach to improve the solution has been given in [29].

2.4.4 LMI-Based Iterative Algorithm of Helmersson

The optimal H_∞-norm approximation can be formulated as an LMI problem. Given $G_n(s) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$, let $G_r(s) = \left[\begin{array}{c|c} \mathbf{A}_r & \mathbf{B}_r \\ \hline \mathbf{C}_r & \mathbf{D}_r \end{array}\right]$ be the optimal r^{th} order approximation of $G_n(s)$. The approximation problem can be formulated as: Find minimum γ and $G_r(s) = \left[\begin{array}{c|c} \mathbf{A}_r & \mathbf{B}_r \\ \hline \mathbf{C}_r & \mathbf{D}_r \end{array}\right]$ such that

$$\begin{bmatrix} A^TP_{11} + P_{11}A & A^TP_{12} + P_{12}A_r & P_{11}B - P_{12}B_r & C^T \\ A_r^TP_{12}^T + P_{12}^TA & A_r^TP_{22} + P_{22}A_r & P_{12}^TB - P_{22}B_r & C_r^T \\ B^TP_{11}^T - B_r^TP_{22}^T & B^TP_{12} - B_r^TP_{22} & -\gamma I & D^T - D_r^T \\ C & C_r & D - D_r & -\gamma I \end{bmatrix} < 0 \qquad (2.13)$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is a symmetric positive definite matrix. The algorithm can be summarized below.

***ALGORITHM* 2.6: LMI-Based Helmersson Algorithm**[47]

Given $G_n(s)$ and an initial estimate of $G_r(s)$,

Step 1: Keeping A_r, B_r fixed, minimize γ subject to the constraint (2.13) with respect to P, C_r, D_r

Step 2: Keeping P_{12}, P_{22} fixed, minimize γ subject to the constraint (2.13) with respect to P_{11}, A_r, B_r, C_r, D_r.

Repeat Steps 1 and 2 until the solution converges.

The Helmersson's algorithm [47] does not yield a convex problem when the order is fixed and the solution is not guaranteed to converge to a global minimum.

2.4.5 Complex Curve Fitting

The basic approach of complex curve fitting is to obtain the frequency response curve of the original system, which is a complex valued function of the frequency, then to find the transfer function of the desired order that best fits the original curve with respect to a criterion. In 1959, Levy [48] introduced and solved a complex curve-fitting algorithm, which minimizes the sum of squares of the error. Santhanan and Koerner [49] proposed an iterative adjustment of the weights to improve the estimate. Whitfield [50] investigated the asymptotic behavior of several iterative adjustment techniques. Spanos [51] proposed an iterative weighted least squares problem that can be used to solve the least maximum amplitude problem. Hakvoort and Van Den Hof [52] considered weighted l_∞ criteria which is summarized as follows.

***ALGORITHM* 2.7: Complex Curve Fitting** [52]

Given $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$, $G_n(\omega)$, a positive real valued weight function $W(\omega)$ for $\omega \in \Omega$ and the desired order r. Find $G_r(\omega) = \frac{n_r(\omega)}{d_r(\omega)}$ by solving the following constrained optimization problem

$$\min_{h, n_r(\omega), d_r(\omega)} \quad h^2 \quad \text{subject to}$$

$$W^2(\omega)|G_n(\omega)d_r(\omega) - n_r(\omega)|^2 \leq h^2|d_r(\omega)|^2$$

The complex curve fitting algorithm transforms the H_∞-norm model reduction problem to a nonlinear constrained optimization problem. The algorithm is directly applicable for both continuous and discrete-time systems. Payne [53], observed that the curve fitting algorithm

tends to give unstable models if high orders are used. For this reason, the approximant is restricted to be stable. Several well-known optimization software can be used to solve the above constrained optimization problem. To solve this problem in a reasonable way, one needs to have a good initial estimate.

2.4.6 Differential Correction Algorithm

The differential correction algorithm was first described by Cheney and Loeb [54]. It was initially developed to obtain rational approximation of real functions. Modified versions appeared in [55], and [56]. The best rational approximation is obtained by iteratively solving a minimization problem having linear objective function with quadratic constraint. Recently the differential correction algorithm has been extended to simultaneously approximating the real and imaginary parts of a complex function [57], [58] and has been applied to the design of stable IIR filters. The basic algorithm is summarized as follows.

***ALGORITHM* 2.8: Differential Correction Algorithm** [57]

Given a complex valued function $f(\omega_k)$, $\omega_k \in \Omega$, desired orders m, n and Δ_m

Step 1: Let $l = 1$ and choose an initial approximation to $f(\omega_k)$, $\frac{P_1}{Q_1} \in \Re_{m,n}$

Step 2: Determine $P_{l+1}(x)$ and $Q_{l+1}(x)$ of degree m, n respectively that minimize

$$\max_{\omega_k} \left\{ \frac{|f(\omega_k)Q_{l+1}(\omega_k) - P_{l+1}(\omega_k)| - \Delta_l Q_l(\omega_k)}{Q_l(\omega_k)} \right\}$$

where

$$\Delta_l = \max_{\omega_k \in \Omega} \left| f(\omega_k) - \frac{P_l(\omega_k)}{Q_l(\omega_k)} \right|$$

and

$$Q_l(\omega_k) \neq 0$$

Step 3: If $\Delta_l \leq \Delta_m$, stop. Otherwise set $l = l + 1$ and go to Step 2.

2.4.7 Iterative Reweighted Least Squares Algorithm

Consider the polynomial approximation problem: Given $A_d(\omega_k)$ for $\omega_k \in \Omega$, find a polynomial $A(\omega_k)$ that minimize

$$\varepsilon = \sum_{k=0}^{N} |A(\omega_k) - A_d(\omega_k)|^p \,. \tag{2.14}$$

It is known that an optimal solution exists and it is unique ([59], Corollary 7.5.4), but there is no simple direct method to obtain the optimal approximation for all $p \neq 2$. The error criteria in (2.14) can be expressed as

$$\varepsilon = \sum_{k=0}^{N} W_k^2 \, |A(\omega_k) - A_d(\omega_k)|^2 \tag{2.15}$$

where

$$W_k^2 = |A(\omega_k) - A_d(\omega_k)|^{p-2} \,. \tag{2.16}$$

If W_k is known, then minimizing (2.14) is equivalent to minimizing (2.15) for which a closed form solution exists. Unfortunately, the weight in (2.15) is not known in advance. The iterative reweighted least squares algorithm (IRWLS) attempts to solve the problem as follows. In the first iteration the problem (2.15) is solved assuming unit weight to obtain an estimate of $A(\omega_k)$ denoted by $\hat{A}(\omega_k)$. The weight is computed using (2.16). If the algorithm is a contraction mapping then it will converge and the limit point is the solution of the l_p approximation

problem. It was shown that there exists a unique $A(\omega_k)$ solving (2.14) but there may exist more than one weight function such that $A(\omega_k)$ is also the optimal solution of (2.15).

The use of IRWLS was first developed by Lawson [60]. His algorithm will be discussed in more detail in the next section.

2.4.8 Lawson's Algorithm

The Lawson's algorithm was proposed in 1961 [60] to solve the l_∞ optimal regression problem by means of solving an iterative weighted l_p approximation problem. The basic problem considered by Lawson is outlined next. Given $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}, F = \{f(\omega_1), \cdots, f(\omega_N)\}$, approximate the function f by a function of the form

$$L(\mathrm{a}, \omega) = \sum_{j=1}^{n} a_j \, \Phi_j(\omega)$$

such that

$$\max_k |f(\omega_k) - L(\mathrm{a}, \omega_k)|$$

is minimum where $\Phi_j(\omega)$ is a Chebyshev set. In the following, a summary of Lawson's algorithm is given.

***ALGORITHM* 2.9: Lawson's Algorithm**

Given $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$, $F = \{f(\omega_1), \cdots, f(\omega_N)\}$ and $\{\Phi_1, \Phi_2, \cdots, \Phi_n\}$

Step 1: Let $l = 1$; $U^l(\omega_k) = 1 \quad \forall \omega_k \in \Omega$

Step 2: Solve the following weighted least squares problem

$$\min_{\mathbf{a}} \left\| U^l(\omega_k) \left[f(\omega_k) - \sum_{j=1}^{n} a_j^l \, \Phi_j(\omega_k) \right] \right\|_2$$

Step 3: Update the weighting function using

$$U^{l+1}(\omega_k) = \frac{U^l(\omega_k) \left| f(\omega_k) - \sum_{j=1}^{n} a_j^l \, \Phi_j(\omega_k) \right|}{\sum_{\omega_k \in \Omega} \left| f(\omega_k) - \sum_{j=1}^{n} a_j^l \, \Phi_j(\omega_k) \right|}$$

Step 4: Stop if $\|\mathbf{a}^l - \mathbf{a}^{l-1}\|$ is within acceptable tolerance. Otherwise set $l = l + 1$ and go to Step 2.

Lawson's algorithm is shown to be slow (linear convergence)[61]. However, it was proved in [60, 62] that in the limit, Lawson's algorithm converges to the optimal solution of the optimal l_∞ regression problem. For faster convergence $U^l(\omega_k)$ should tend to zero as rapidly as possible except for extremal points. It was pointed out in [62] that it is important to avoid setting $U^l(\omega_k)$ to zero at least in the early stages, since this may cause missing the best approximation. On the other hand, the speed of convergence depends on the rate of decay of $U^l(\omega_k)$. One possible acceleration technique is to modify step 3 of the Lawson's algorithm to

Step 3': Update the weighting function using

$$U^{l+1}(\omega_k) = \frac{\left| U^l(\omega_k) \right|^s \cdot \left| f(\omega_k) - \sum_{j=1}^{n} a_j^l \, \Phi_j(\omega_k) \right|^t}{\sum_{\omega_i \in \Omega} \left| f(\omega_k) - \sum_{j=1}^{n} a_j^l \, \Phi_j(\omega_k) \right|}$$

where s and t are positive constants. The acceleration techniques were suggested to improve the speed of convergence and it was observed that they may converge much faster than the original Lawson's algorithm. However, the convergence of these techniques is not guaranteed.

The main limitation is that the approximating function need to be expressed as a linear combination of some basis functions. As a consequence, polynomial approximation (FIR) can be solved directly by Lawson's algorithm.

***ALGORITHM* 2.10: FIR Approximation Using Lawson's Algorithm**

Given $G(z) \in \Re H_\infty$, $\Omega = \{\omega_1, \omega_2, ..., \omega_N\}$, the number of iterations $l_{\max}$ and the desired order r.

Step 1: Discretize $G_n(z)$ on N points $z = e^{j\omega_k}$. Let $l = 1$, $U^l(\omega_k) = 1 \quad \forall \omega_k \in \Omega$.

Step 2: Compute $n_r(e^{j\omega_k}) = b_0 + b_1 e^{-j\omega_k} + ... + b_r e^{-jr\omega_k}$ by minimizing

$$\min_{b_0, b_1, ..., b_r} \sum_{k=1}^{N} \left| \left(G(e^{j\omega_k}) - n_r(e^{j\omega_k}) \right) U^l(e^{j\omega_k}) \right|^2$$

Step 3: Update $U^l(e^{j\omega k})$ using

$$U^{l+1}(e^{j\omega_k}) = \left(G(e^{j\omega_k}) - n_r(e^{j\omega_k}) \right) U^l(e^{j\omega_k}) / \alpha$$

where α is a scaling factor.

Step 4: If $l = l_{\max}$, stop. Otherwise set $l = l + 1$ and go to Step 2.

Rational functions can not be expressed as a linear combination of a basis set. Because of this restriction, rational approximations can not be obtained by Lawson's algorithm. To overcome this problem, one may attempt the following modified problem. Instead of minimizing

$$\left\| G(e^{j\omega_k}) - \frac{n_r(e^{j\omega_k})}{d_r(e^{j\omega_k})} \right\|_2$$

which is a nonlinear approximation problem, one may try to minimize

$$\left\| G(e^{j\omega_k}) d_r(e^{j\omega_k}) - n_r(e^{j\omega_k}) \right\|_2$$

which can be solved using the following modified version of the Lawson's algorithm.

***ALGORITHM* 2.11: Approximation by the Lawson-Loeb Approach**

Given $G(z) \in \Re H_\infty$, the number of iterations $l_{\max}$ and the desired order r.

Step 1: Discretize $G_n(z)$ on N points $z_k = e^{j\omega_k}$ for $\omega_k \in \Omega$. Let $l = 1$, $U^l(\omega_k) = 1$.

Step 2: Compute $d_r(e^{j\omega_k})$ and $n_r(e^{j\omega_k})$ by minimizing the

$$\min_{b_0, b_1, \ldots, b_r, a_1, a_2, \ldots, a_r} \sum_{k=1}^{N} \left| \left(G(e^{j\omega_k}) d_r(e^{j\omega_k}) - n_r(e^{j\omega_k}) \right) U^l(e^{j\omega_k}) \right|^2, \tag{2.17}$$

Step 3: Update $U^l(e^{j\omega k})$ using

$$U^{l+1}(e^{j\omega_k}) = \left(G(e^{j\omega_k})\, d_r(e^{j\omega_k}) - n_r(e^{j\omega_k}) \right) U^l(e^{j\omega_k})/\alpha$$

where α is a scaling factor,

Step 4: If $l = l_{\max}$, stop. Otherwise set $l = l + 1$ and go to Step 2.

Unfortunately most of the time the optimal solution of (2.17) does not lead to a good solution to the original problem (2.3). A more serious problem is that the solution is often unstable. The Lawson's algorithm has been generalized to solve the H_∞-One Block Problem via the weighted H_2 optimization problem [63].

2.4.9 Ellacott-Williams Algorithm

The Ellacott-Williams algorithm was developed in [64], and [65]. The algorithm uses Lawson's algorithm to come up with estimates of the numerator and denominator polynomials and then solve an optimization problem to obtain the final estimate. The basic algorithm is given below.

***ALGORITHM* 2.12: Ellacott-Williams Algorithm** [66]

Given G_n, numerator and denominator orders m, r and a scalar T.

Step 1: Choose $n_r^0(z) \in P_r$, $d_r^0(z) \in P_r$, $d_r^0(z) \neq 0 \; \forall \, z$ inside the unit disk.

Step 2: Find $\delta_n^k \in P_m, \delta_d^k \in P_r$ by Lawson's algorithm that minimize

$$\left\| G_n - \frac{n_r^k}{d_r^k} - \frac{d_r^k \delta_n^k - n_r^k \delta_d^k}{[d_r^k]^2} \right\|_\infty$$

Step 3 : Set $d_r^{k+1} = d_r^k + t\ \delta_d^k$, $n_r^{k+1} = n_r^k + t\ \delta_n^k$ where

$$\left\| G_n - \frac{p_{k+1}^*}{q_{k+1}^*} \right\|_\infty \leq \left\| G_n - \frac{p_k^* + t\,\delta\, p_k}{q_{k+1}^* + \delta\, p_k} \right\|_\infty \qquad \forall t \in [0, T]$$

Step 4: Stop if the error is below the desirable level. Otherwise set $l = l + 1$ and go to Step 2.

The algorithm was shown to be slow and converges to either a local best approximation or a saddle point. The algorithm has been used in the design of IIR filters. Chen and Park [66] proposed a modified version of the algorithm to improve the speed of convergence. The

Chen-Park algorithm [66] uses linear programming techniques which the authors claimed to reduce computation time by about one half of the Ellacott-Williams Algorithm. Lee and Chen proposed a generalization of the Ellacott-Williams algorithm in [67].

2.5 Summary

In this chapter, the H_∞-norm approximation problem is defined and existing algorithms to solve it are summarized. So far, there is no globally accepted algorithm to solve the H_∞-norm approximation problem.

The balanced model truncation and optimal Hankel norm approximation are widely used to give approximations to the optimal H_∞-norm reduced order models. Neither method is directly related to the optimal H_∞-norm model reduction problem, but both are widely used because they often give solutions that are close to the optimal solution. In addition, upper and lower bounds on the deviation form the optimal solution can be obtained. Both algorithms are based on state-space models. The Hankel approximation results in a tighter bound compared to balanced approximation but the reduced order model may be unstable.

It is difficult to obtain the optimal H_∞-norm approximation using the state-space characterization of optimal approximation, but the suboptimal approximations are obtained more easily. The iterative LMI-based suboptimal method seems to be more promising. The LMI-based algorithm of Helmersson is not guaranteed to converge.

In the complex curve fitting, the approximation problem is transformed to a nonlinear constrained optimization problem. To be able to solve it in a reasonable time a good initial solution is needed. The procedure often results in an unstable approximant and therefore the

reduced order model should be constrained to be stable.

The Lawson's algorithm was proved to converge to the optimal l_∞ regression problem. However, it is restricted to being a linear combination of some basis functions. As a consequence, FIR approximation can be obtained using Lawson's algorithm. A modified version known as Loeb-Lawson algorithm gives rational approximation but the solution may be far from the solution of the original problem and the reduced order model may be unstable. The Ellacott-Williams algorithm was shown to be slow and it may converge to a saddle point.

Several other algorithms to solve the optimal H_∞-model reduction problem have been proposed. They include: complex curve fitting, differential correction and iterative reweighted least squares methods. However, so far, there is no known general and efficient solution procedure.

Chapter 3

NEW H_∞ APPROXIMATION ALGORITHMS

3.1 Introduction

In this chapter, several new algorithms to solve the H_∞-norm approximation problem are proposed. The proposed algorithms are iterative and are classified into two groups: time-domain and frequency-domain algorithms. These algorithms use the transfer function representation. One advantage of using this representation is the considerable reduction in the number of the parameters involved compared to the state-space representations.

In Section 3.2 motivations for the new approach are given. Identification-based algorithms are developed in Section 3.3, the frequency-domain algorithms are given in Section 3.4, approximation of continuous systems is discussed in Section 3.5 and the approximation of multivariable systems is considered in Section 3.6. Examples are given in Section 3.7. Evaluation of the proposed algorithms is given in Section 3.8 and extensions to constrained approximation and

approximation of unstable systems are given in Sections 3.10 and 3.11 respectively. Finally the frequency selection approach is discussed in Section 3.12.

3.2 Motivation

The proposed algorithms follow the general approach of the iterative reweighted least squares algorithm discussed in Section 2.4.7. The H_∞-norm approximation is obtained by successively solving simpler weighted approximation problems.

Consider the H_∞-norm model reduction problem

$$\min_{G_r \in \mathfrak{R}H_\infty} \|G_n - G_r\|_\infty . \tag{3.1}$$

It will be shown later that, more than one weight $W(z)$ exist, such that the optimal solution of (3.1) is also the optimal solution of

$$\min_{G_r \in \mathfrak{R}H_\infty} \|W\,(G_n - G_r)\|_2 . \tag{3.2}$$

It will also be shown that there exists $W(z)$ such that the solution of (3.1) is also a solution to the following simpler problem.

$$\min_{d_r, n_r \in P_r} \|W\,(G_n\, d_r - n_r)\|_2 \tag{3.3}$$

Consider the problem of finding the weight $W(z)$, such that

$$\|G_n - G_r\|_\infty = \|W\,(G_n - G_r)\|_2 \tag{3.4}$$

where G_n and G_r are known. There may exist more than one weight which satisfies the equation (3.4). The following lemma gives two possible weights.

Lemma 6 *Given G_n and G_r both in $\Re L_\infty$, equation (3.4) is satisfied if the weight $W(z)$ is*

$$W\left(e^{j\omega}\right) = \begin{cases} \sqrt{\frac{\pi}{2\epsilon}} & \text{if } |\omega - \omega_0| < \epsilon \text{ or } |\omega + \omega_0| < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

where ω_0 is the frequency at which the maximum of $|G_n - G_r|$ is attained, that is

$$\|G_n - G_r\|_\infty = \left|G_n\left(e^{j\omega_0}\right) - G_r\left(e^{j\omega_0}\right)\right|.$$

An alternative choice of $W(z)$ is

$$W\left(e^{j\omega}\right) = \begin{cases} \frac{\|G_n - G_r\|_\infty}{|G_n(e^{j\omega}) - G_r(e^{j\omega})|} & \text{if } \left|G_n\left(e^{j\omega}\right) - G_r\left(e^{j\omega}\right)\right| \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. By direct substitution of $W\left(e^{j\omega}\right)$ in the expression of $\|W\left(G_n - G_r\right)\|_2$. ■

The significance of the above lemma is that it shows the possibility of converting the H_∞-norm model reduction problem into a weighted H_2-norm model reduction problem. The H_2-norm model reduction problem in (3.2) is a nonlinear optimization problem that may be difficult to solve. In the following lemma two weights that make (3.1) equivalent to (3.3) are derived.

Lemma 7 *Given G_n and $G_r \in \Re L_\infty$, if one selects the weight $W(z)$ as*

$$W\left(e^{j\omega}\right) = \begin{cases} d_r(e^{j\omega})\sqrt{\frac{\pi}{2\epsilon}} & \text{if } |\omega - \omega_0| < \epsilon \text{ or } |\omega + \omega_0| < \epsilon \\ 0 & \text{otherwise} \end{cases}, \tag{3.5}$$

where ω_0 is the same as in Lemma 6, or selects the weight as

$$W\left(e^{j\omega}\right) = \begin{cases} \frac{\|G_n - G_r\|_\infty}{|G_n(e^{j\omega})\, d_r(e^{j\omega}) - n_r(e^{j\omega})|} & \text{if } \left|G_n\left(e^{j\omega}\right) d_r\left(e^{j\omega}\right) - n_r\left(e^{j\omega}\right)\right| \neq 0 \\ 0 & \text{otherwise} \end{cases}, \qquad (3.6)$$

then

$$\|G_n - G_r\|_\infty = \|W\left(G_n\, d_r - n_r\right)\|_2$$

is satisfied.

Proof. By direct substitution of $W\left(e^{j\omega}\right)$ in the expression of $\|W\left(G_n\, d_r - n_r\right)\|_2$. ■

So far, two classes of weights are used to show the equivalence of the H_∞ and the H_2 model reduction problems. A major limitation is that the frequency dependent weights are functions of the optimal reduced order model that is to be determined. The general strategy that will be used in this work is as follows. First, an initial estimate of $G_r(z)$ is obtained by solving the unweighted least squares model reduction problem and this estimate is used to obtain an estimate of the weight $W(z)$ which will be used to obtain an improved estimate of $G_r(z)$ and the procedure is repeated till $G_r(z)$ and $W(z)$ converge to some limit values.

In the following, only the weight in (3.5) will be used. Unless the error is perfectly circular, the weight is zero except at the points for which the error is at its maximum. There can be at most $n + r + 1$ such points and it is expected that the peak frequencies change in the initial iterations but gradually converge to some limit points. One possible way of obtaining the weights is as follows.

$$W(e^{j\omega}) = \frac{\left|G_n\left(e^{j\omega}\right) - G_r\left(e^{j\omega}\right)\right|^m}{\alpha}$$

where m is a large positive integer and α is a scaling factor which is used to avoid numerical problems but has no effect since the solution of the H_2 approximation problem depends on the relative values of the weights. This choice makes $W(e^{j\omega})$ almost zero for most frequencies and non-zero at finite frequency intervals. A modified version is to be used where the weighting function at the l^{th} iteration is given by

$$W^l(e^{j\omega}) = \frac{\left|G_n\left(e^{j\omega}\right) - G_r\left(e^{j\omega}\right)\right|^l}{\alpha}.$$

The motivation for this is that it is expected that peak frequencies change in the first couple of iterations and hence a smaller exponent is used. As the number of iteration steps increases the non-zero weight is limited to smaller and smaller intervals. An alternative approach is to define the weight based on the error and the weight of the previous iteration as follows

$$W^{l+1}(e^{j\omega}) = \frac{W^l(e^{j\omega})\left|G_n\left(e^{j\omega}\right) - G_r^l\left(e^{j\omega}\right)\right|^2}{\alpha}. \tag{3.7}$$

The formula in (3.7) is the one we will be using in this thesis. This updating scheme is very similar to the one used in Lawson's algorithm.

3.3 Identification Based H_∞-Norm Approximation Techniques

System identification is used to determine a mathematical model from a given model set according to a set of input-output data and some optimality criteria. Given a set of input-output pairs $\{u(t), y(t)\}$, one can use an identification technique, such as the least squares, to obtain a model of the desired order that fits the data. Furthermore, one can obtain an r^{th} order

approximate model using identification techniques as follows:

1. Select an input sequence $u(t)$.

2. Apply the input sequence $u(t)$ to the original system G_n and observe the system response $y(t)$.

3. Using an identification method, obtain the r^{th} order model $G_r(z)$.

The model $G_r(z)$ is optimal with respect to the objective function defined by the identification method, which is not necessarily the optimal H_∞-norm approximation of $G_n(z)$. Recall that the identified model depends on the input sequence and there may exist an input sequence that results in $G_r(z)$ which is the optimal H_∞-norm approximation of $G_n(z)$. Unfortunately there is no known way of obtaining such an input sequence.

Given a set of N input-output pairs $\{u(t), y(t)\}$ for $t \in [1, N]$ which satisfies the input-output relationship

$$A(q^{-1})y(t) = B(q^{-1})u(t) + e(t) \tag{3.8}$$

with

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{na} q^{-na}$$

and

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_{nb} q^{-nb}.$$

The sequence $e(t)$ is a set of independent and identically distributed random variables and q^{-1} is the backward shift operator. Let g_r and g_n be the impulse responses of G_r and G_n respectively and let U and Y be the Z-transform of the sequences u and y respectively. The output $y(t)$

can be generated by

$$y(t) = g_n(t) * u(t)$$

where "$*$" is the convolution operator.

Now consider the H_∞-norm approximation problem. From the definition of $\| \cdot \|_\infty$, we have

$$\min_{G_r} \|G_n - G_r\|_\infty = \min_{G_r} \sup_{u \in \ell_2} \frac{\|(G_n - G_r)U\|_2}{\|u\|_2}.$$

Given a transfer function $G_n(z)$, one can select an input sequence $u(t)$ for $t \in [1, N]$ and compute the corresponding output sequence. It is required to obtain a G_r in the following form

$$G_r(z, \theta) = \frac{B_r(z^{-1})}{A_r(z^{-1})} = \frac{b_0 + b_1 z^{-1} + \ \ldots \ b_r z^{-r}}{1 + a_1 z^{-1} + \ \ldots \ a_r z^{-r}}$$

where

$$\theta = [a_1 \ \ldots \ a_r \ \ b_0 \ \ b_1 \ \ \ldots \ b_r]^T .$$

The optimal H_∞-norm approximation problem is to obtain an estimate of θ that minimizes the H_∞-norm of the error. In the following sections we present two identification based algorithms: the output error, OE, and the least squares, LS, methods.

3.3.1 Output Error Algorithm

In this algorithm, we try to fit the data assuming an output error model of the form

$$y(t) = \frac{B_r(q^{-1})}{A_r(q^{-1})} u(t) + e(t)$$

where all the variable are defined in an analogous way as in (3.8). The output error estimate of θ, denoted by θ_{OE} is obtained by minimizing

$$\epsilon_{OE}(t,\theta) = y(t) - \frac{B_r(q^{-1})}{A_r(q^{-1})}u(t). \tag{3.9}$$

There is no closed form expression for the output error estimates. The output error identification problem is

$$\min_{G_r \in H_\infty} \frac{\|y - y_r\|_2}{\|u\|_2}$$

By Parsevals' theorem,

$$\min_{G_r \in H_\infty} \frac{\|y - y_r\|_2}{\|u\|_2} = \min_{G_r \in H_\infty} \frac{\|(G - G_r)U\|_2}{\|u\|_2}.$$

It is desirable to select $u \in \ell_2$ which is a persistently exciting signal to achieve

$$\frac{\|(G - G_r)U\|_2}{\|u\|_2} \rightarrow \min_{G_r} \|G_n - G_r\|_\infty .$$

An input sequence satisfying the above condition is called the optimal excitation. The proposed algorithm is iteratively modifying $u(t)$ to get the optimal excitation, which is motivated by the Lawson's algorithm. Recall that in Laswon's algorithm the weight at the l^{th} iteration is updated using

$$U^{l+1}\left(e^{j\omega_k}\right) = E^l\left(e^{j\omega_k}\right)\ U^l\left(e^{j\omega_k}\right)\ /\alpha$$

where $E^l\left(e^{j\omega_k}\right)$ is the error at the l^{th} iteration which is defined as

$$E^l\left(e^{j\omega_k}\right) = G\left(e^{j\omega_k}\right) - G_r^l\left(e^{j\omega_k}\right)$$

and α is a scaling factor. In the OE-based H_∞ -norm approximation algorithm, the new excitation is obtained by multiplying the previous excitation by the error in the previous iteration. In time-domain, the new excitation is obtained by convolving the previous error with the previous input.

ALGORITHM 3.1: OE-Based H_∞ Approximation [71]

Given the impulse response $g(t)$, the desired order r, and the number of iterations $l_{\max}$.

Step 1: *Let $l = 1$ and select an initial excitation sequence u^1 as a white noise random sequence.*

Step 2: *Compute $y^l = g * u^l$*

Step 3: *Obtain $G_r^l(\theta^l, z)$ by solving the OE-identification problem*

$$\theta^l = \arg\min_{\theta} \sum_{t=1}^{N} \left|\epsilon_{OE}\left(t, \theta\right)\right|^2$$

where $\epsilon_{OE}\left(t, \theta\right)$ is defined in (3.9).

Step 4: *Update the input signal according to $u^{l+1} = \left(y^l - g_r^l * u^l\right)/\alpha$ where $\alpha = \sum_{t=1}^{N} \left|\epsilon_{OE}\left(t, \theta\right)\right|$*

Step 5: *If $l = l_{\max}$, stop. Otherwise set $l = l + 1$ and go to Step 2.*

Remark 1 *In identifying linear systems, the magnitude of the input signal has little effect on the outcome of the identification process and one needs to avoid extreme cases that lead to numerical problems. The value of α in Step 4 is used to prevent numerical problems. When the algorithm is terminated, $G_r(z)$ is expected to be close to the optimal H_∞ approximation.*

3.3.2 Least Squares Algorithm

The least squares estimation is a well known technique. The least squares estimator is a linear estimator with quadratic objective function and therefore the solution always converges to the global minimum. In this section a brief summary of the least squares algorithm and some of its properties is given. Further information is available in a large number of references such as [69, 70].

Let the parameter vector θ be defined as

$$\theta = [a_1\ a_2\ \ldots\ a_r\ b_0\ b_1\ \ldots\ b_r]^T$$

and the regressor Φ be defined as

$$\Phi(t) = [-y(t-1)\ -y(t-2)\ \ldots -y(t-r)\ u(t)\ u(t-1)\ \ldots\ u(t-r)]^T$$

The model structure (3.8) can be expressed as

$$y(t) = \Phi^T(t)\theta + e(t)$$

The least squares estimate of θ is obtained by minimizing the loss function

$$V_{LS}(\theta) = \sum_{t=1}^{N} \varepsilon_{LS}^2(t,\theta)$$

where the residual ε is defined as

$$\varepsilon_{LS}(t,\theta) = y(t) - \Phi^T(t)\theta. \tag{3.10}$$

The least squares estimate θ is obtained using the following formula:

$$\theta = \left[\sum_{t=1}^{N} \Phi(t)\ \Phi^T(t)\right]^{-1} \left[\sum_{t=1}^{N} \Phi(t)\ y(t)\right]. \tag{3.11}$$

An alternative way is to express this in matrix form. Define

$$\mathbf{\Phi} = \begin{bmatrix} \Phi(1) \\ \Phi(2) \\ \vdots \\ \Phi(N) \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}$$

Then, the unknown parameter vector can be determined using

$$\theta = \left[\ \mathbf{\Phi}^T\mathbf{\Phi}\right]^{-1} \mathbf{\Phi}^T\ \mathbf{Y}. \tag{3.12}$$

The computation of θ can be done using (3.12). More reliable numerical techniques such as Cholesky factorization and singular value decomposition can be used to obtain better estimates. It is well known that in the presence of measurement noise, the least squares produces biased and inconsistent estimates. Despite these disadvantages, the least squares method is widely used. A major reason for this is that it is a noniterative scheme and hence computationally very fast.

The basic approach to least squares H_∞-norm approximation is to use the weighted least squares method to obtain estimates of G_n and iteratively modify the weights to obtain new estimates. The algorithm is given next.

ALGORITHM 3.2: LS H_∞ Model Reduction [71]

Given the impulse response $g(t)$, the number of iterations $l_{\max}$, and the desired order r.

Step 1: *Let $l = 1$ and select an initial excitation sequence u^1 as a white noise random sequence.*

Step 2: *Compute $y^l = g * u^l$*

Step 3:*Obtain $G_r^l(\theta^l, z)$ by solving the LS-identification problem*

$$\theta^l = \arg\min_{\theta} \sum_{t=1}^{N} |\epsilon_{LS}(t, \theta)|^2$$

where $\epsilon_{LS}(t, \theta)$ is defined in (3.10).

Step 4: *Update the excitation $u^{l+1} = \left(y^l - g_r^l * u^l\right)/\alpha$ where $\alpha = \sum_{t=1}^{N} |\epsilon_{LS}(t, \theta)|$*

Step 5: *If $l = l_{\max}$, stop. Otherwise set $l = l + 1$* and go to Step 2.

3.3.3 Weighted H_∞-Norm Approximation

The problem considered here is to find G_r that minimizes $\|W(G_n - G_r)\|_\infty$ where $W(z)$ is a frequency dependent weight which can be used to emphasize error in certain frequency bands.

Given $G_n(z)$ or its impulse response $g_n(t)$, and the weighting function $W(z)$ or its impulse response $w(t)$, the weighted approximation error is obtained by convolving the error with the impulse response of the weight. The weighted output error is defined as

$$\epsilon_{WOE}(t, \theta) = (y(t) - y_r(t)) * w(t) = (g_n(t) - g_r(t)) * u(t) * w(t). \tag{3.13}$$

Let $y_w(t)$ and $u_w(t)$ be defined as

$$y_w(t) = y(t) * w(t), \qquad \text{and} \quad u_w(t) = u(t) * w(t), \tag{3.14}$$

then the weighted error in (3.13) can be expressed as

$$\epsilon_{wOE}(t,\theta) = \; y_w(t) \; - g_r(t) * u_w(t).$$

The main difference between the weighted and the unweighted cases is that the input and output should be filtered through the weight.

ALGORITHM 3.3: Weighted OE-Based H_∞ Model Reduction

Given the impulse responses $g(t)$, $w(t)$, the number of iterations $l_{\max}$, and the desired order r,

Step 1: *Let $l = 1$ and select an initial excitation sequence u^1 as a white noise random sequence..*

Step 2: *Compute $u_w^l = w(t) * u^l(t)$ and $y_w^l = g(t) * u_w^l(t)$*

Step 3: *Obtain $G_r^l(\theta^l, z)$ by solving the OE-identification problem*

$$\theta^l = \arg\min_{\theta} \sum_{t=1}^{N} |\epsilon_{WOE}(t,\theta)|^2$$

where $\epsilon_{WOE}(t,\theta)$ is defined in (3.13)

Step 4: *Update the excitation $u^{l+1} = \left(y^l - g_r^l * u^l\right)/\alpha$*

Step 5: *If $l = l_{\max}$, stop. Otherwise set $l = l + 1$* and go to Step 2.

The Algorithm 3.3 is a generalization of Algorithm 3.1 where the input and output signals are filtered by the weight in each iteration.

The weighted H_∞-norm model reduction based on LS is developed next. The algorithm

minimizes the weighted LS error which is defined as

$$\epsilon_{WLS}(t,\theta) = (y(t) - \Phi\theta) * w(t) \tag{3.15}$$

Defining Φ_w as

$$\Phi_w(t) = [-y_w(t-1) \ -y_w(t-2) \ \ldots -y_w(t-na) \ u_w(t) \ u_w(t-1) \ \ldots \ u_w(t-nb)]^T,$$

then the weighted LS error is given by

$$\epsilon_{WLS}(t,\theta) = y_w(t) - \Phi_w(t)\,\theta. \tag{3.16}$$

ALGORITHM 3.4: Weighted LS-Based H_∞-Norm Model Reduction

Given the impulse responses $g(t)$ and $w(t)$, the number of iterations $l_{\max}$ and the desired order r,

Step 1: *Let* $l = 1$ and select an initial excitation sequence u^1 as a white noise random sequence.

Step 2: *Compute* $u_w^l(t) = u^l(t) * w(t)$ *and* $y_w^l(t) = y^l(t) * w(t)$

Step 3: *Obtain* $G_r^l(\theta^l, z)$ *by solving the LS-identification problem*

$$\theta^l = \arg\min_{\theta} \sum_{t=1}^{N} |\epsilon_{WLS}(t,\theta)|^2,$$

where $\epsilon_{WLS}(t,\theta)$ is defined in (3.16)

Step 4: *Update the excitation* $u^{l+1} = \left(y^l - g_r^l * u^l\right)/\alpha$

Step 5: *If* $l = l_{\max}$, *stop. Otherwise set* $l = l + 1$ and go to Step 2.

It is clear that the above algorithm is a generalization of Algorithm 3.2.

3.3.4 The H_∞−Norm Approximation with Spectrum Smoothing

The stability of the approximation algorithm may be improved by smoothing of the frequency spectrum. Several techniques can be used to perform smoothing. In this section the algorithms 3.1 and 3.2 are modified to perform spectrum smoothing. One such spectrum smoothing technique is done by time series modeling of the error. In each iteration an r^{th} order model is obtained which is used to obtain an estimate of the error spectrum. The error is modeled by an autoregressive (AR) model as follows

$$C(q^{-1})\,\epsilon\,(t,\theta) = v(t)$$

where

$$C(q^{-1}) = 1 + c_1 q^{-1} + c_2 q^{-2} + \cdots + c_m q^{-m}$$

and $v(t)$ is an i.i.d. random sequence. The order m is chosen to obtain a reasonable fit of the data. Once $C(q^{-1})$ is estimated then the new excitation can be computed using

$$C(q^{-1})\,u^{l+1}(t) = u^1(t).$$

In the following an OE-based approximation with smoothing is presented.

ALGORITHM **3.5: OE-based H_∞ Model Reduction with Smoothing**

Given the impulse response $g(t)$, the number of iterations $l_{\max}$, m and the desired order r.

Step 1: *Let* $l = 1$ and select an initial excitation sequence u^1 as a white noise random sequence.

Step 2: *Compute* $y^l = g * u^l$

Step 3: *Obtain* $G_r^l(\theta^l, z)$ *by solving the OE-identification problem*

$$\theta^l = \arg\min_{\theta} \sum_{t=1}^{N} |\epsilon_{OE}(t, \theta)|^2$$

where $\epsilon_{OE}(t, \theta)$ is defined in (3.9).

Step 4: *Compute* $\epsilon_{OE}(t, \theta)$ then fit an AR model through $\epsilon_{OE}(t, \theta^l)$. The assumed model has the following form

$$C(q^{-1})\epsilon_{OE}\left(t, \theta^l\right) = v(t)$$

where

$$C(q^{-1}) = 1 + c_1 q^{-1} + c_2 q^{-2} + \cdots + c_m q^{-m}$$

and $v(t)$ is an i.i.d. random sequence.

Step 5: *Update the excitation using* $C(q^{-1})\, u^{l+1}(t) = u^0(t)$.

Step 6: *If* $l = l_{\max}$, *stop. Otherwise set* $l = l + 1$ and go to Step 2.

Simulation studies indicated that selecting $m = \min(5, r + 2)$ gives a satisfactory result. The LS-based approximation can be modified in a similar way.

3.3.5 Using FFT in H_∞−Norm Approximation

The identification based algorithms can be easily converted to frequency domain ones, so that one can use the numerically efficient FFT techniques to reduce computation time. Another mo-

tivation for using the frequency-domain technique is to perform approximation using frequency-domain data. Such an approach allows design of digital filters whose ideal frequency response cannot be represented by rational transfer functions. In time-domain, the output corresponding to an input sequence can be computed by convolving the sequence with the impulse response of the system. The convolution in time-domain is equivalent to multiplication in the frequency-domain. Thus it is more efficient to compute the output in the frequency-domain. In the following, the notation DFT and IDFT stand for N-point Discrete Fourier Transform and its inverse.

The modified LS algorithm is given below.

ALGORITHM 3.6: H_∞ Model Reduction using LS and FFT

Given $G(e^{j\frac{2\pi k}{N}})$ *for* $k \in [1, N]$, *the number of iterations* l_{max} *and the desired order* r.

Step 1: *Let* $l = 1$ *and select an initial excitation sequence* u^1 *as a white noise random sequence. Compute* $U^1 = DFT(u^1)$.

Step 2: *Compute* $Y^l(k) = G(k)U^l(k)$ *for* $k \in [1, N]$, and $y^l(t) = IDFT(Y^l)$.

Step 3: *Obtain* $G_r^l(e^{j\frac{2\pi k}{N}}, \theta^l)$ *by solving the LS-identification problem*

$$\theta^l = \arg\min_{\theta} \sum_{t=1}^{N} |\epsilon_{LS}(t, \theta)|^2$$

where $\epsilon_{LS}(t, \theta)$ is defined in (3.10).

Step 4 : *Compute* $U^{l+1}(k) = (G(e^{j\frac{2\pi k}{N}}) - G_r^l(e^{j\frac{2\pi k}{N}}, \theta^l))U^l$

Step 5 : *If* $l = l_{\max}$, *stop. Otherwise set* $l = l + 1$ and go to Step 2.

3.4 Frequency Domain H_∞-Norm Approximation Algorithm

In the frequency domain H_∞-norm approximation algorithms, a transfer function, which is a complex valued function, is to be approximated by a lower order one. In implementing the algorithms, samples of the transfer function at N frequencies are used. This makes it possible to approximate systems based on frequency response measurements rather than the transfer function. This is desirable in some applications such as filter design problems where the ideal filter cannot be represented as a rational transfer function.

Consider the linear system $G(z)$; Assume the input to be $U(z)$ and then the output $Y(z)$ is computed from

$$Y(z) = G(z)U(z).$$

Let G_r be an approximation of $G(z)$. The output of $G_r(z)$ is

$$Y_r(z) = G_r(z)U(z).$$

The output error can be computed as

$$Y(z) - Y_r(z) = G(z)U(z) - G_r(z)U(z) = (G(z) - G_r(z))U(z). \tag{3.17}$$

If G and G_r are available then by definition

$$\|G - G_r\|_\infty = \sup_{U \in L_2} \|(G - G_r)U\|_2$$

and there exists $U^*(z)$ such that

$$\|G - G_r\|_\infty = \|(G - G_r)U^*\|_2.$$

It is well known that $U^*(z)$ corresponding to the worst case input has the properties that it is equal to zero except at the frequencies at which

$$|(G(e^{j\omega}) - G_r(e^{j\omega}))| = \|G - G_r\|_\infty.$$

Now for a given $U(z)$, consider the following H_2 approximation problem

$$\min_{G_r \in H_\infty} \|(G - G_r)U\|_2. \tag{3.18}$$

If $U^*(z)$ exists and is known then the H_∞-norm approximation problem is solved by solving (3.18). There are two problems in this approach. First, the question of existence of $U^*(z)$, and if it exists, how to find it. Second, solving (3.18), which is a nonlinear optimization problem, can be difficult to solve. Therefore, instead of using the output error in (3.18), the equation error will be used. The equation error minimization problem is

$$\min_{n_r(z), d_r(z)} \|(d_r G - n_r)U\|_2.$$

The output error and the equation error criteria are related as follows

$$\|(d_r G - n_r)U\|_2 = \|(G - G_r)d_r U\|_2.$$

The equation error criterion is similar to the output error criteria except that $d_r(e^{j\omega_k})$ appears as a multiplier multiplying $U(e^{j\omega_k})$. It was mentioned earlier that the optimal excitation is zero except at a few frequencies; Therefore the effect of $d_r(e^{j\omega_k})$ is simply to scale the excitation at these frequencies.

3.4.1 Frequency Domain Identification

The problem here is to obtain a rational transfer function using a set of N samples of the frequency response of the transfer function. Given $G(k) = G(e^{j\omega_k})$ for $\omega_k \in \Omega$, it is desirable to find G_r having the form

$$G_r(e^{j\omega_k}) = \frac{n_r(e^{j\omega_k})}{d_r(e^{j\omega_k})} = \frac{b_0 + b_1 e^{j\omega_k} + \cdots + b_r e^{jr\omega_k}}{1 + a_1 e^{j\omega_k} + \cdots + a_r e^{jr\omega_k}} \tag{3.19}$$

Such that $\|G - G_r\|_2$ is minimized. The LS technique cannot be used directly because G_r is rational and therefore cannot be expressed as a linear combination of a basis function. There exist no Φ such that $G_r(k) = \Phi(k)\ \theta$. Consider the equation error form

$$\min_{n_r(z), d_r(z)} \|d_r G - n_r\|_2. \tag{3.20}$$

Equation (3.20) can be expressed as

$$\min_{\theta} \sum_{k=1}^{N} |G(k) - \Phi(k)\theta|^2$$

where

$$\Phi(k) = \left[-G\left(e^{j\omega_k}\right) e^{j\frac{2\pi k}{N}} \; - G\left(e^{j\omega_k}\right) e^{j2\frac{2\pi k}{N}} \; \cdots \; - G\left(e^{j\omega_k}\right) e^{jr\frac{2\pi k}{N}} \;\; 1 \;\; e^{j\frac{2\pi k}{N}} \;\; e^{j2\frac{2\pi k}{N}} \;\; \cdots \;\; e^{jr\frac{2\pi k}{N}} \right] \tag{3.21}$$

and

$$\theta = [a_1 \;\; a_2 \;\; \cdots \; a_r \;\; b_0 \;\; b_1 \;\; \cdots \; b_r \,]^T . \tag{3.22}$$

Using the above notation, the following weighted approximation problem

$$\min_{n_r(z), d_r(z)} \; \| \left(d_r G - n_r\right) U \|_2 \tag{3.23}$$

can be expressed as

$$\min_{\theta} \sum_{k=1}^{N} |G\left(k\right) U(k) - \Phi(k)\theta \; U(k)|^2$$

where Φ and θ are defined in (3.21) and (3.22) respectively.

3.4.2 Frequency Domain Algorithm

Now we are ready to present the new H_∞-norm model reduction algorithm. The algorithm uses samples of the transfer function on N frequencies. This makes it possible to approximate systems based on frequency response measurements rather than the transfer function.

***ALGORITHM* 3.7: Frequency Domain H_∞ Model Reduction Algorithm**

Given $G(z) \in RH_\infty$, $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$, the number of iterations l_{max} and the desired order r.

Step 1: Discretize $G_n(z)$ on the points $z = e^{j\omega_k}$ for $\omega_k \in \Omega$ and set $U^1(e^{j\omega_k}) = 1$ for all $k \in [1, N]$.

Step 2: Solve for G_r having the structure (3.19) from

$$\min_{b_0,b_1,\ldots,b_r,a_1,a_2,\ldots,a_r} \|G_n(e^{j\omega_k})d_r(e^{j\omega_k}) - n_r(e^{j\omega_k}))U^l(e^{j\omega_k})\|_2$$

Step 3: Update $U(e^{j\omega_k})$

$$U^{l+1}(e^{j\omega_k}) = \left|\left(G_n(e^{j\omega_k}) - G_r^l(e^{j\omega_k})\right)\ U^l(e^{j\omega_k})\right| \ /\alpha$$

Step 4: If $l = l_{max}$, stop. Otherwise set $l = l + 1$ and go to step 2.

Remark 2 *If*

$$\max_{\omega}\left|\left(G_n(e^{j\omega}) - \ G_r^l(e^{j\omega})\right)\right| - \min_{\omega}\left|\left(G_n(e^{j\omega}) - \ G_r^l(e^{j\omega})\right)\right|$$

is very small then the error curve is nearly circular. Using Lemma 2, one can conclude that the obtained solution is optimal or very close to the optimal solution. Unfortunately this will only occur in a few cases.

The weighted case is considered in the following algorithm.

ALGORITHM 3.8: Frequency domain Weighted H_∞ Model reduction

Given $G_n(z) \in R\mathbf{H}_\infty$, $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$, the number of iterations l_{max} and the desired order r.

Step 1: Discretize $G_n(z)$ and $W(z)$ on the points $z = e^{j\omega_k}$ for $\omega_k \in \Omega$ and set $U^1(e^{j\omega_k}) = 1$ for all $k \in [1, N]$.

Step 2: Solve for G_r having the structure (3.19) from

$$\min_{b_0,b_1,\ldots,b_r,a_1,a_2,\ldots,a_r} \|G(e^{j\omega_k})d_r(e^{j\omega_k}) - n_r(e^{j\omega_k}))U^l(e^{j\omega_k})W(e^{j\omega_k})\|_2$$

Step 3: Update $U(e^{j\omega_k})$ using

$$U^{l+1}(e^{j\omega_k}) = \left|\left(G_n(e^{j\omega_k}) - G_r(e^{j\omega_k})\right) W(e^{j\omega_k})U^l(e^{j\omega_k})\right| \ /\alpha$$

Step 4: if $l = l_{max}$, stop. Otherwise set $l = l + 1$ and go to Step 2.

3.5 H_∞-Norm Approximation of Continuous-Time Systems

The frequency domain algorithm extends directly to continuous time systems without any modifications to the algorithm. Unlike the discrete time case where the frequency varies in the range $[0, 2\pi]$, the frequency range is infinite for the continuous case. The first step in applying the approximation is to select the range of interest. One can take $\omega \in [0, \omega_u) + \infty$ where ω_u is large enough.

Approximation of continuous time can be done indirectly using the discrete-time algorithms. In order to do this, use bilinear transformation to convert the continuous-time problem into a discrete-time problem, apply model reduction and then convert the reduced order model back to continuous time. The following transformation is used:

$$s = \frac{z-1}{z+1} \Longleftrightarrow z = \frac{1+s}{1-s}$$

Given a continuous-time state-space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, the bilinear transformation leads to the discrete-time model

$$\left[\begin{array}{c|c} (A+I)(I-A)^{-1} & \sqrt{2}\,(I-A)^{-1}B \\ \hline \sqrt{2}C(I-A)^{-1} & D+C(I-A)^{-1}B \end{array}\right]$$

Similarly, given the discrete-time system $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, the continuous-time equivalent is

$$\left[\begin{array}{c|c} (A+I)^{-1}\,(A-I) & \sqrt{2}\,(I+A)^{-1}B \\ \hline \sqrt{2}C(I+A)^{-1} & D-C(I+A)^{-1}B \end{array}\right]$$

Furthermore, it can be shown [11] that the Gramians of a discrete-time system are equal to those of the continuous-time equivalent. As a result the HSV are identical for both cases.

3.6 Approximation of Multivariable Systems

In this section, the approximation of Multivariable systems is considered. Given a $p \times q$ transfer function G_n, it is required to approximate it with a simpler model G_r having the form

$$G_r(z) = A_r^{-1}(z)B_r(z) \tag{3.24}$$

where $A_r(z)$ and $B_r(z)$ are polynomial matrices of dimension $p \times p$ and $p \times q$ respectively. The elements of $A_r(z)$ and $B_r(z)$ are assumed to have the general structure

$$A_{ij}(z) = a_{i,j,0} + a_{i,j,1}z + a_{i,j,2}z^2 + \cdots + a_{i,j,\nu_{ij}}z^{\nu_{ij}}$$

$$B_{ij}(z) = b_{i,j,0} + b_{i,j,1}z + b_{i,j,2}z^2 + \cdots + b_{i,j,\mu_{ij}}z^{\mu_{ij}}.$$

A major source of difficulty of approximating the MIMO system is the need to specify $p^2 + pq$ structural indices (i.e., ν_{ij}, μ_{ij}). The quality of the approximation depends on the selection of these indices and one may need to scan all possible combinations until the best choice is obtained.

The problem of selecting the right structural indices is an important problem in the identification of MIMO systems and several special cases have been considered in current literature [70]. In this work two special cases will be considered.

3.6.1 The Full Polynomial Form

This is the simplest form with only two structural indices to be selected. The model is given by 3.24 with

$$A(z) = I + A_1 z + \cdots + A_{n_a} z^{n_a}$$

$$B(z) = B_0 + B_1 z + \cdots + B_{n_b} z^{n_b}.$$

The unknown parameters can be grouped as

$$\theta = [A_1\, A_2 \cdots A_{n_a}\, B_0\, B_1 \cdots B_{n_b}]$$

The regressor is defined as

$$\phi^T(t) = \left[-y^T(t-1), -y^T(t-2), \cdots, -y^T(t-n_a)\,, u^T(t),\, u^T(t-1), u^T(t-n_b)\right].$$

The system model is given by

$$y^T(t) = \phi^T(t)\,\theta$$

Writing the system model in matrix form we have.

$$Y = \Phi\theta \quad \text{with } Y = \begin{bmatrix} y^T(1) \\ y^T(2) \\ \vdots \\ y^T(N) \end{bmatrix}, \text{ and } \Phi = \begin{bmatrix} \phi^T(1) \\ \phi^T(2) \\ \vdots \\ \phi^T(N) \end{bmatrix}$$

The least squares estimate is given by

$$\theta = \left(\Phi^T\Phi\right)^\dagger \Phi^T Y$$

Although the above form is a simple extension of the SISO case, this form is not canonical. There exist linear systems of the form (3.24) that cannot be uniquely parameterized by θ. Note

also that $\Phi^T\Phi$ may not be invertible and therefore the pseudo-inverse is used.

3.6.2 Diagonal Form

The diagonal form is a canonical representation. It is more general than the full polynomial form with $2p$ structural indices. The model is given by (3.24) with

$$A(z,\theta) = \begin{bmatrix} a_1\ (z,\theta) & & 0 \\ & \ddots & \\ 0 & & a_p\ (z,\theta) \end{bmatrix}, \qquad B(z,\theta) = \begin{bmatrix} \mathbf{b}_1\ (z,\theta) \\ \vdots \\ \mathbf{b}_p\ (z,\theta) \end{bmatrix}$$

where

$$a_i(z) = 1 + a_{i,1}z + \cdots + a_{i,na_i}\, z^{na_i}$$

$$\mathbf{b}_i(z) = b_{i,0} + b_{i,1}\, z + \cdots + b_{i,nb_i}\, z^{nb_i}.$$

There are $2p$ structural parameters to be specified compared to two in the full polynomial form and $p^2 + pq$ for the general form, while the McMillan degree is less than or equal to $\sum_{i=1}^{p} na_i$.

ALGORITHM 3.9: Frequency Domain Weighted H_∞ Approximation (MIMO)

Given $G_n(z) \in R\mathbf{H}_\infty$, the desired order r, the number of iterations $l_{\max}$, the structural indices and N.

Step 1: Discretize $G_n(z)$, $W_i(z)$ and $W_o(z)$ on N points $z = e^{j\omega_k}$. Set $l = 1$, $U^l(e^{j\omega_k}) = 1$.

Step 2: Solve for G_r from

$$\min_{\theta} \ \|W_o(e^{j\omega_k}\ )\left(G(e^{j\omega_k})A_r(e^{j\omega_k},\theta) - B_r(e^{j\omega_k},\theta)\right)W_i(e^{j\omega_k}\ )U^l(e^{j\omega_k})\|_2$$

Step 3:Update $U^l(e^{j\omega_k})$

$$U^{l+1}(e^{j\omega_k}) = \ \sigma_{\max}\left(W_o(e^{j\omega_k}\ )\left(G_n(e^{j\omega_k}) - G_r(e^{j\omega_k})\right)W_i(e^{j\omega_k})U^l(e^{j\omega_k})\right)\ /\alpha$$

Step 4: *If* $l = l_{\max}$, *stop. Otherwise set* $l = l + 1$ *and go to Step 2.*

3.7 Examples

In this section several examples are given to illustrate the approximation algorithms presented and to compare them with other well known approximation techniques.

3.7.1 Example 3.1: A Fifth Order Discrete-Time System

Consider the following fifth order system.

$$G(z) = \frac{0.0014z^5 - 0.0215z^4 + 0.0533z^3 + 0.1978z^2 - 1.1463z}{z^5 - 1.1463z^4 + 0.1978z^3 + 0.0533z^2 - 0.0215z + 0.0014}$$

The Hankel singular values are $7.1343, 2.4044, 1.0456, 0.6471$ and 0.0016. The proposed approximation techniques are applied to the system. The error norm $\|G - G_r\|_\infty$ is given for $r = 1, 2, 3$ and 4 in Table 3.1.

TABLE 3.1: H_∞-Norm of the Error for Example 3.1

degree of approximation	1	2	3	4
Algorithm 3.1	2.7657	1.2707	0.8912	0.0199
Algorithm 3.2	2.6624	1.1180	0.7451	0.0017
Algorithm 3.6	2.7033	1.1963	0.7504	0.0162
Algorithm 3.7	2.6624	1.1180	0.7450	0.0016
Hankel Approximation	2.9262	1.6500	0.6476	0.0016
Balanced Approximation	4.8213	2.1048	1.2973	0.0032
Lower bound (HSV)	2.4044	1.0456	0.6471	0.0016

The reduced order models produced by Algorithm 3.7 are given below.

$$
\begin{aligned}
G_1 &= \frac{0.6131z - 1.2637}{z - 0.9506}, G_2 = \frac{-0.1185z^2 + 0.4288z - 0.6232}{z^2 - 1.7121z + 0.7442} \\
G_3 &= \frac{-0.0141z^3 + 0.0128z^2 + 0.1243z - 0.5667}{z^3 - 1.6431z^2 + 0.8189z - 0.1329}, \\
G_4 &= \frac{0.0014z^4 - 0.0215z^3 + 0.0533z^2 + 0.1978z - 1.1463}{z^4 - 1.1463z^3 + 0.1981z^2 + 0.0514z - 0.0185}.
\end{aligned}
$$

In this example, it is clear that the least square based algorithms perform better than other algorithms except for the case $r = 3$ where the Hankel approximation results in a smaller approximation error.

3.7.2 Example 3.2: A Sixth Order Continuous-Time System

Consider the following 6th order continuous time system:

$$
G(s) = \frac{(s-2)^6}{(s^2 + 0.5s + 1)^2(s+1)^2}
$$

The above algorithms were applied to obtain reduced order approximation.

TABLE 3.2: H_∞-Norm of the Error for Example 3.2

Degree of Approximation	1	2	3	4
Algorithm 3.2	211.3236	101.0922	101.0559	31.4110
Algorithm 3.7	211.3230	100.8754	100.7762	31.4036
Hankel Approximation	351.7961	166.3895	109.8171	32.5014
Balanced Approximation	320.7280	202.6748	186.9867	59.6783
Lower bound (HSV)	193.0026	89.5016	84.2125	29.7294

Here also, both Algorithms 3.2 and 3.7 perform better than the algorithms based on the Hankel approximation or balanced reduction. The resulting approximation error is close to the lower bound which is not necessarily achievable.

3.7.3 Example 3.3: A Weighted Approximation Example

Now consider the following weighted approximation problem. Let $G(s)$ be

$$G(s) = \frac{s^2 + 0.2s + 1.01}{s^2 + 0.2s + 4.04} \quad \frac{s^2 + 0.2s + 9.01}{s^2 + 0.2s + 16.02}$$

Find the r^{th} order weighted approximation with the following weight.

$$W = \frac{(s-1)^2}{s^2 - 0.2s + 1}$$

This problem has been studied in [72] and [38]. Algorithm 3.8 is used to obtain the second and the third order approximations

$$G_2 = \frac{0.7988s^2 + 1.9076s + 2.8343}{s^2 + 0.3221s + 16.4489}$$

$$G_3 = \frac{2.5449s^3 + 2.7671s^2 + 33.1183s - 6.5111}{s^3 + 4.9777s^2 + 16.9316s + 74.9863}$$

Table 3.3 gives the H_∞-Norm on the Error of reduced order approximation for the second and third order approximation with different methods.

TABLE 3.3: H_∞-Norm of the Error for Example 3.3

Degree of Approximation r	2	3
Algorithm 3.3	4.629	3.835
Algorithm 3.8	4.2018	3.5055
Liu & Anderson [72]	20.08	11.94
Zhou(Algorithm I) [38]	4.827	8.20
Zhou(Algorithm II) [38]	4.822	3.946
Weighted Balanced [32]	5.128	3.946
Lower Bound (HSV)	2.7037	2.5267

From the above table one can see that the proposed algorithms out perform the compared algorithms.

3.7.4 Example 3.4: A Seventh Order Continuous-Time System

Consider the following system

$$G(s) = \frac{0.05\left(s^7 + 801s^6 + 1024s^5 + 599s^4 + 451s^3 + 119s^2 + 49s + 5.55\right)}{s^7 + 12.6s^6 + 53.48s^5 + 90.94s^4 + 71.83s^3 + 27.22s^2 + 4.75s + 0.3}$$

It is required to obtain a weighted approximation. The weight is taken as

$$W(s) = G^{-1}(s).$$

TABLE 3.4: H_∞-Norm of the Error for Example 3.4

Degree of Approximation r	2	3	4	5
Algorithm 3.2	1.000	0.7160	0.6493	0.0026
Algorithm 3.8	1.000	0.7159	0.6488	0.0026
Liu &Anderson[72]	7.09	0.8417	0.6106	0.0026
Algorithm I of [38]	7.09	0.8103	0.5949	0.0026
Algorithm II of [38]	1.00	0.7197	0.5594	0.0026
Balanced Truncation	15.4947	4.2022	2.0820	0.0052
Lower Bound (HSV)	0.9734	0.7146	0.5584	0.0026

From the above table, it is clear that Algorithm 3.8 gives better results except for the case $r = 4$, which is not as good as the others.

3.7.5 Example 3.5: Weighted Approximation of Sixth Order System

This weighted approximation example was studied by Anderson [72] and Zhou [73]. The original transfer function is given by

$$G(s) = \frac{1}{s^6 + 3.8637s^5 + 7.4641s^4 + 9.1416s^3 + 7.4641s^2 + 3.8637s + 1}$$

It is required to obtain G_r such that

$$\|W_\alpha (G_n - G_r)\|_\infty$$

The weighting function is

$$W_\alpha = \frac{(s-1)^2}{s^2 - 2\alpha s + 1}$$

Note that in our approach, only the magnitude of the weight is important. Fourth order approximation will be obtained for the cases $\alpha = 0.1$ and 0.01. The fourth order approximation are obtained using Algorithm 3.8 with

$$G_{0.1}(s) = \frac{0.0212s^4 - 0.0942\ s^3 + 0.2528\ s^2 - 0.4308\ s + 0.4355}{s^4 + \ 1.5055s^3 + \ 1.9200s^2 + \ 1.2157s + 0.4450}$$

$$G_{0.01}(s) = \frac{0.0234s^4 - 0.1007\ s^3 + 0.2649\ s^2 - 0.4436s + 0.4426}{s^4 + \ 1.5126s^3 + \ 1.9381s^2 + \ 1.2280s + 0.4533}.$$

For comparison, the table below shows the weighted approximation error with the methods reported in [72] and [73].

TABLE 3.5: H_∞-Norm of the Error for Example 3.5

α	0.1	0.01
Algorithm 3.8	0.0225	0.0249
Liu & Anderson[72]	0.0272	0.0860
Algorithm I of [38]	0.0248	0.0448
Algorithm II of [38]	0.0225	0.0256
Balanced Truncation [32]	0.0509	0.0584
Lower Bound (HSV)	0.0219	0.0248

It is clear from the above table that Algorithm 3.8 is as good as the reported algorithms or better.

3.7.6 Example 3.6: An Eighth Order Continuous System

In this example, we apply Algorithm 3.7 to obtain reduced order approximations of the following 8th order system:

$$G(s) = 10\frac{(s-1)^2}{(s^2+s+1)^4}.$$

The above system has been studied in [29] and two algorithms based on Optimal characterization have been used. In the following table, a summary of the result is given.

Table 3.6: H_∞-Norm of the Error for Example 3.6

Degree of Approximant	1	2	3	4	5	6	7
Algorithm 3.7	23.5015	14.2499	10.4389	3.1177	0.73488	0.0940	0.0134
Algorithm 1[29]	23.3726	15.6117	11.8938	3.6498	0.7981	0.0983	0.0134
Algorithm 2[29]	23.3100	14.1454	10.3367	3.1075	0.7344	0.0940	0.0134
Hankel Approximation	30.4932	21.7677	11.2919	3.1707	0.7362	0.0940	0.0134
Lower Bound (HSV)	20.9943	13.1886	9.04484	2.9549	0.7304	0.0938	0.0134

From Table 3.6, Algorithm 3.7 is comparable to Algorithm 2 of [29].

3.7.7 Example 3.7: A Randomly Generated Example

Consider the following randomly generated example. It is a 2×2 system represented as

$$G_n(z) = A^{-1}(z)B(z)$$

where A and B are given by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.9830 & 0.4001 \\ 0.5527 & 0.1988 \end{bmatrix} z + \begin{bmatrix} 0.6252 & 0.3759 \\ 0.7334 & 0.0099 \end{bmatrix} z^2 + \begin{bmatrix} 0.4199 & 0.7939 \\ 0.7537 & 0.9200 \end{bmatrix} z^3$$

$$B = \begin{bmatrix} 0.1556 & 0.4225 \\ 0.1911 & 0.8560 \end{bmatrix} + \begin{bmatrix} 0.4902 & 0.4608 \\ 0.8159 & 0.4574 \end{bmatrix} z + \begin{bmatrix} 0.4507 & 0.9016 \\ 0.4122 & 0.0056 \end{bmatrix} z^2 + \begin{bmatrix} 0.2974 & 0.6932 \\ 0.0492 & 0.6501 \end{bmatrix} z^3$$

Reduced order approximation $G_r(z) = A_r^{-1}(z)B_r(z)$ were obtained for $r = 1$ and 2.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.9696 & -1.2058 \\ 0.1609 & 0.8774 \end{bmatrix} z$$

$$B_1 = \begin{bmatrix} 3.2252 & -5.9269 \\ 1.1028 & -0.0496 \end{bmatrix} + \begin{bmatrix} 2.6732 & -4.5681 \\ 1.1306 & 0.6378 \end{bmatrix} z$$

with minimum error $\|G_n - G_1\|_\infty = 9.5571$.

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2.7453 & -1.9599 \\ 0.4411 & 1.0316 \end{bmatrix} z + \begin{bmatrix} 1.6097 & -2.8139 \\ 0.2986 & -0.2063 \end{bmatrix} z^2$$

$$B_2 = \begin{bmatrix} -4.1769 & 0.4803 \\ -0.2617 & 1.2705 \end{bmatrix} + \begin{bmatrix} -5.0346 & -5.3164 \\ -0.0934 & 0.8755 \end{bmatrix} z + \begin{bmatrix} -2.7524 & -3.6527 \\ -0.2704 & -0.2926 \end{bmatrix} z^2$$

with minimum error $\|G_n - G_2\|_\infty = 4.5037$.

3.7.8 Example 3.8: A Sixth Order Continuous-Time System

Consider the following 6^{th} order continuous-time system:

$$G(s) = \frac{s\,(-s+1.5)\,(-s+2.5)\,(-s+4.5)\,(-s+5.5)}{(s+1)\,(s+2)\,(s+3)\,(s+4)\,(s+5)\,(s+6)}.$$

This system was studied in [121]. Algorithm 3.7 was used to obtain the reduced order model for different orders. The results are summarized in Table 3.7.

TABLE 3.7: H_∞-Norm of Approximation Error for Example 3.8

Degree of Approximant	1	2	3	4
Algorithm 3.7	0.81014	0.55216	0.29283	0.10560
Method 1 of [121]	0.81001	0.56334	0.29821	0.10631
Hankel Approximation	1.0818	0.6843	0.33634	0.11140
Lower Bound (HSV)	0.73853	0.49290	0.26328	0.09913

The reduced order models are given by

$$
\begin{aligned}
G_1 &= \frac{-1.1495s + 15.7553}{s + 160.2151} \\
G_2 &= \frac{-0.4969s^2 + 5.255s - 28.558}{s^2 + 6.4781s + 208.4278} \\
G_3 &= \frac{-1.1815s^3 + 17.80s^2 - 74.24s + 174.29}{s^3 + 37.658s^2 + 137.49s + 1422.9} \\
G_4 &= \frac{-0.9038s^4 + 12.776s^3 - 67.059s^2 + 137.77s - 125.503}{s^4 + 16.534s^3 + 181.664s^2 + 467.760s + 1762.30}
\end{aligned}
$$

From Table 3.7, the Algorithm 3.7 is comparable or better than other algorithms.

Now we consider the weighted approximation problem with the frequency weighting

$$W(s) = \frac{10}{s + 10}.$$

Algorithm 3.8 was used to obtain reduced order models and the results are summarized in Table 3.8.

TABLE 3.8: Weighted H_∞-Norm of Approximation Error for Example 3.8

Degree of Approximant	1	2	3	4
Algorithm 3.8	0.5796	0.45233	0.27518	0.10217
Method 1 of [121]	0.58471	0.49804	0.27853	0.10416
Lower Bound (WHSV)	0.39034	0.2572	0.18398	0.08857

and the reduced order models are given by

$$
\begin{aligned}
G_1 &= \frac{0.5788s + 1.1268}{s + 50.2099} \\
G_2 &= \frac{0.1802s + 0.1915s - 2.0918}{s^2 + 1.7035s + 58.0364} \\
G_3 &= \frac{-0.9447s^3 + 20.1932s^2 - 77.6616s + 204.0470}{s + 51.6987s + 136.7158s + 1729.9} \\
G_4 &= \frac{-0.7069s^4 + 10.32s^3 - 54.81s^2 + 112.42s - 103.26}{s^4 + 12.8529s^3 + 160.0s^2 + 365.797s + 1502.9}
\end{aligned}
$$

From Table 3.8, the Algorithm 3.8 is better than compared algorithms.

3.8 Evaluation of the Proposed Algorithms

In this section the proposed algorithms will be evaluated. Important factors that are investigated are: the computation time, the minimum achievable error and convergence properties.

The proposed algorithms are iterative techniques with two main decision variables in each method. They are the number of iterations $l_{\max}$ and N the length of the input sequence for the time domain algorithms or the number of samples of the frequency response for the frequency

domain algorithms. In the following we will study the effects of both $l_{\max}$ and N or the studied properties.

3.8.1 Minimum Achievable Error

Since for most of the problems considered, the optimal reduced order model is not known, the minimal achievable error is compare with the lower bound provided by the HSV and with other the error of other existing techniques. As expected, the proposed algorithms are, in general, much better than the suboptimal approximations obtained by balanced truncation and Hankel norm approximations.

It was observed that in general, increasing both $l_{\max}$ and N results in lower approximation error. However, in most cases very good approximations can be obtained using relatively smaller values of N. Using the system in Example 3.1, the effect of using different N for Algorithm 3.2 and 3.7 was investigated and the result is shown in Figure 3-2 and Figure 3-1.

The convergence properties is mainly affected by N. The behavior of different methods as a function of N were investigated for LS-based algorithms. Typical behavior is shown in Figures 3.1 and 3.2. In general, increasing N will result in a smoother curve.

It was also observed that most of the reduction of the approximation error is obtained in the first few iterations. In many cases it is monotonically decreasing, but there are cases where the error curve is not monotonic. For almost all the cases we tried, $l_{\max}$ in the range 30 to 50 was sufficient.

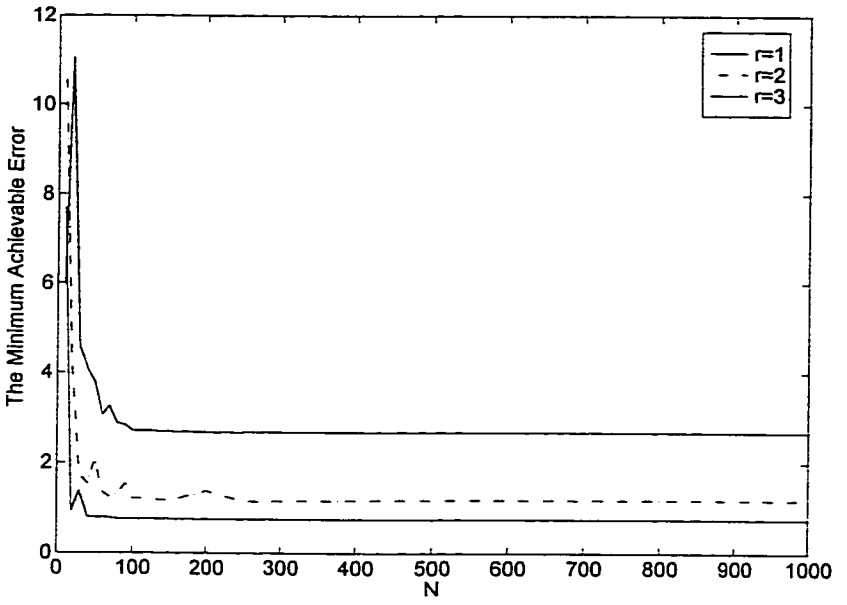

Figure 3-1: Effect of N on the Minimum Achievable Error-Algorithm 3.2

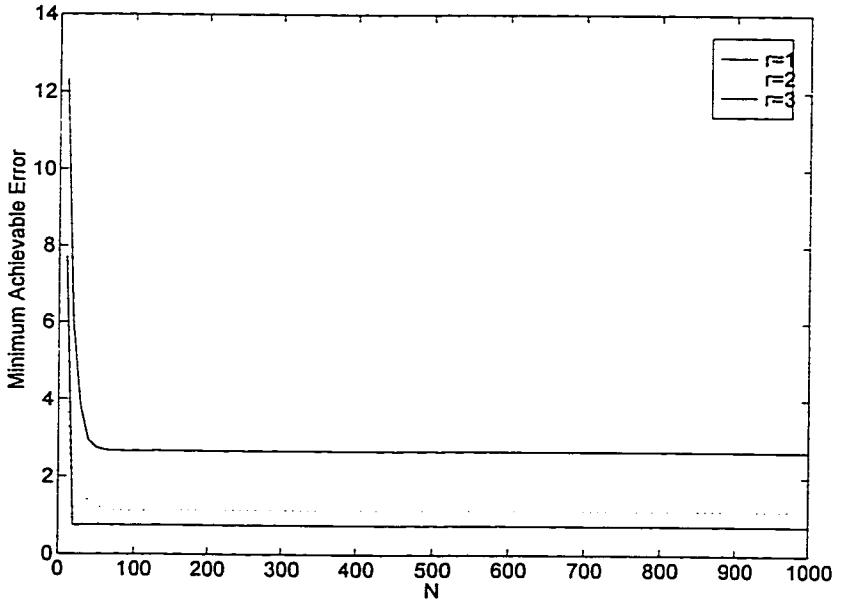

Figure 3-2: Effect of N on the Minimum Achievable Error-Algorithm 3.7

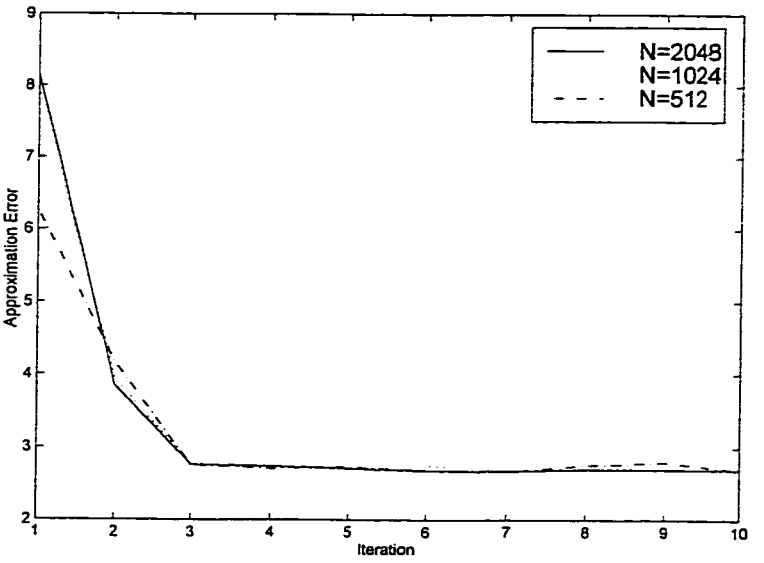

Figure 3-3: Typical Behavior of Algorithm 3.2

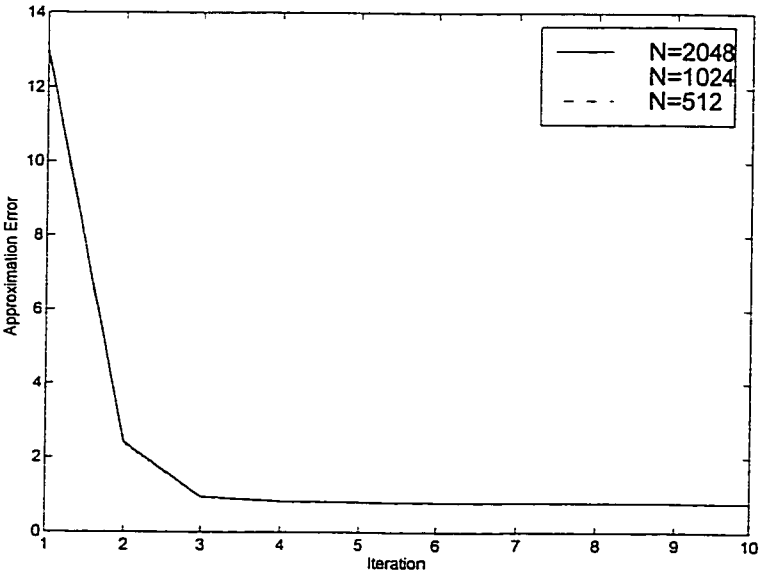

Figure 3-4: Typical Behavior of Algorithm 3.7

3.8.2 Computation Time

In this section we compare the computation time for different algorithms. In the table below the average computation time per iteration is listed for each of the methods. The table below shows the time (in seconds) needed for obtaining a second order approximation of the system used in Example 3.1 averaged over 100 iterations.

Table 3.9: Computation Time for Different Orders

Algorithm	N=32	N=256	N=1024
3.1	0.329	0.4096	0.7965
3.2	0.1922	0.1977	0.2587
3.7	0.0176	0.0527	0.1719

An important note here is that a significant portion of the time is needed for computing the magnitude of the error in each iteration. It is clear that the frequency domain algorithm is much faster than other algorithms while the OE method is the slowest.

3.9 Improving the Performance

In this section we investigate some modifications designed to improve the performance of the proposed algorithms. In studying the behavior of the frequency domain algorithms, we have observed that there are two important factors that affect the convergence speed, namely, the number of frequency samples N and the way the weights are updated. In general, increasing N leads to an improvement in the convergence properties. An idea that was proposed to accelerate Lawson's algorithm is used to improve the convergence by giving higher weights to the frequencies at which larger errors occur. The modified algorithm is given below.

Algorithm 3.10: Accelerated H_∞ Model Reduction Algorithm

All the steps are identical to those of Algorithm 3.7 except for Step 3 which is replaced by

Step 3': Update $U(e^{j\omega_k})$ using

$$U^{l+1}(e^{j\omega_k}) = \left|(G_n(e^{j\omega_k}) - G_r(e^{j\omega_k}))\right|^\nu \Big/ \left|U^l(e^{j\omega_k})\right| \Big/ \alpha$$

where $\nu > 0$ is a specified constant. Usually ν is selected to be more than 1 for higher weights.

Note that the first iteration will produce the LS estimate and the effect of the algorithm starts to appear at the second iteration. Through the use of several examples, the following observations are obtained. When small ν is used, the convergence of the algorithm is slow and when ν is very large the behavior of the algorithm becomes oscillatory. This can be explained by the fact that larger values of ν lead to more emphasis on the frequencies at which larger errors occur which may improve the approximation in the second iteration but, in general, will also cause poor behavior at later iterations. A typical plot of such behavior is given in Figure 3-5. It shows the effect of different acceleration factors for the fourth order approximation of the system in Example 3.1.

In some examples, it was observed that larger values of ν result in a better approximation at the second iteration. This can be explained as follows. The LS estimate obtained in the first iteration, is not equi-ripple in general and therefore the use of high values of ν leads to emphasis being placed on one or few frequency points while the remaining frequencies are ignored. This causes the poor behavior observed in later iterations.

When ν exceeds a certain value, the numerical error dominates and the performance dete-

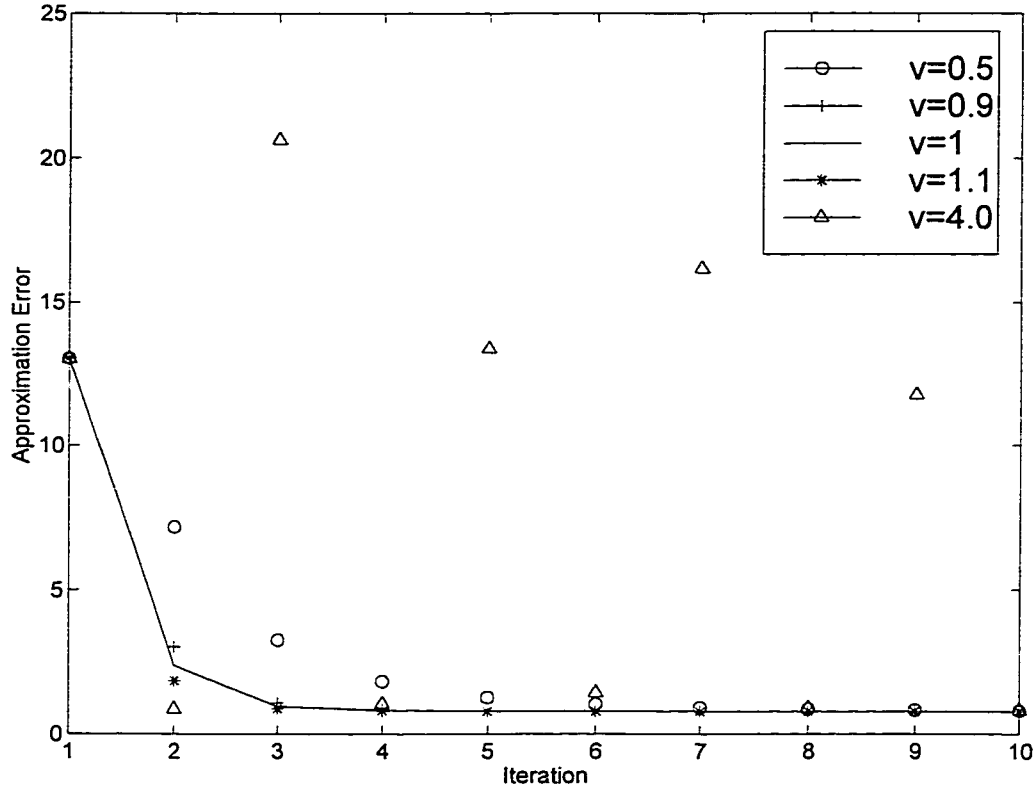

Figure 3-5: Effect of the Acceleration Factor

riorates. Typically a good choice of ν is between 1 and 2.

For the fifth order system in Example 3.1, it was observed that for large acceleration constants, the best result is obtained in the second iteration and the remaining iterations provide no improvement. This can be explained by the fact that large values of the power emphasize the frequency at which the maximum error occurs in the first iteration.

3.9.1 Example 3.9: Effect of the Acceleration Factor

Consider the fifth order system in Example 3.1. Third order reduced order models are obtained using different values of ν in the range $[0.5, 4.0]$ are given. The result is summarized in Figure 3-5. From the figure, one can observe that good performance is obtained for acceleration factors that are close to one.

3.9.2 Reduction of Computation

In the approximation algorithms, it was observed that the convergence properties are improved by using large values of N. This leads to an increase in the computation time. In the frequency domain algorithms, the computation time can be considerably reduced using the following observation. It is well known that the optimal input $U(e^{jw})$ is zero except at finite number points(at most $2r + 2$) which are known as extremal frequencies. It was observed that the frequencies at which the peaks occur are almost unchanged after the first few iterations.

3.10 Constrained H_∞-Norm Approximation

In some applications it is required to pre-specify some of the poles and/or zeros of the approximant. A classical example of such constraint is to fix the denominator of the approximant and

try to find the numerator polynomial. In such a case, the problem becomes a linear approximation problem. In approximating an unstable system, a well known approach is to retain the anti-stable part exactly in the approximant. The constraint approximation can be stated as follows.

Definition 6 *Given $G(z), G_f(z)$ and the desired order r. Find G_r of the order r that minimizes*

$$\|G_n - G_r G_f\|_\infty$$

It is assumed that $G_f(z)$ satisfies

$$0 < |G_f(e^{j\omega})| < \infty$$

The above problem can be converted to a weighted approximation problem as follows.

$$\min_{G_r} \|G_n - G_r G_f\|_\infty = \min_{G_r} \left\| G_f \left(G_n G_f^{-1} - G_r \right) \right\|_\infty$$

The above problem can be directly solved by Algorithm 3.7. An alternative approach that could be used to solve the problem would be to view the prespecified transfer function G_f as a filter that proceeds G_r. This approach can be used to derive the following identification based algorithm.

ALGORITHM 3.11:LS Based Constrained H_∞ Approximation

Given the impulse responses g and g_f, the desired order r and the number of iterations $l_{\max}$.

Step 1: *Let $l = 1$, select an initial excitation sequence u^1 as a white noise random sequence.*

Step 2: *Compute* $y^l = g * u^l, y_f^l = g_f * u^l$

Step 3: *Obtain* $G_r^l(\theta^l, z)$ *by solving the LS-identification problem*

$$\theta^l = \arg\min_{\theta} \sum_{k=1}^{N} |\epsilon_{LS}(t,\theta)|^2$$

where $\epsilon_{LS}(t,\theta)$ is defined in (3.10).

Step 4: *Update the excitation* $u^{l+1} = \left(y^l - g_r^l * y_f^l\right)/\alpha$ *where* $\alpha = \sum_{t=1}^{N} |\epsilon_{LS}(t,\theta)|$

Step 5: *If* $l = l_{\max}$, *stop. Otherwise set* $l = l + 1$ *and go to Step 2.*

3.11 Approximation of Unstable Systems

The approximation techniques discussed so far can be used to approximate stable systems. There are three common ways to approximate an unstable system. The first approach was proposed by Glover [11]. It basically copies the anti-stable part exactly and approximates the stable part. Given the original system G_n, factor it into the sum of a stable factor and an unstable factor as follows.

$$G_n(z) = [G_n]_+ + [G_n]_-$$

where $[G_n]_+$ is stable and $[G_n]_-$ is unstable. The approximation technique discussed earlier can be applied to the stable factor $[G_n]_+$ to obtain G_{+r}. The approximation of G_n is then given by

$$G_r(z) = G_{+r} + [G_n]_-.$$

Copying the unstable part exactly may not be optimal. Glover [11] proposed a second approach which tries to improve the approximation by modifying the unstable part based on

the error made in the approximation of the stable part. Although this might be expected to be better than the first algorithm, there is no guarantee that it will be better. In fact, in many cases the first method leads to smaller error.

The third approach [74] is the coprime factorization approach.

Definition 7 *Let $M, N \in RH_\infty$, have the same number of rows, they are left coprime if and only if there exists $U, V \in RH_\infty$ such that $MV - NU = I$.*

Definition 8 *The pair $M, N \in RH_\infty$, constitute left coprime factorization of G if and only if*

1. *M is square and nonsingular,*
2. *$G = M^{-1}N$,*
3. *N and M are left coprime.*

Definition 9 *The pair $M, N \in RH_\infty$, constitute a normalized left coprime factorization of G if and only if N and M are left coprime factors of G and $NN^* + MM^* = I$ for all $z = e^{j\omega}$, and $\omega \in R$.*

The transfer function to be approximated is expressed in terms of the coprime factors.

$$G(z) = N(z)D^{-1}(z) = \hat{D}^{-1}(z)\hat{N}(z)$$

D and N are often normalized. Note that both D and N are stable irrespective of G. The approximation in the coprime factor approach is to approximate D by D_r and N by N_r and the approximate G_r is given by

$$G_r(z) = N_r(z)D_r^{-1}(z).$$

The number of unstable poles of the original and the approximant may not be the same. In some applications this is desirable. If it is required that they have the same number of unstable poles, then a simple test can be used to check that [74].

3.11.1 Example 3.10: A Third Order Unstable System

Consider the system studied in [74]. The transfer function is given by

$$G(s) = \frac{500s + 3400}{s^2 + 505s + 2500} \frac{1000}{s - 50}$$

This system has two stable poles at -5 and -500 and one unstable pole at 50. In this example Algorithm 3.7 is used to obtain second order approximation. The result is compared with the approximate models obtained in [74] and reported in Table 3.10.

TABLE 3.10: Approximation of a Third Order Unstable System.

Method	Approximating Transfer Function	Error $\|G - G_r\|_\infty$
Glover I	$\frac{245}{s+242} \frac{1000}{s-50}$	0.34
All-pass	$\frac{306}{s+327} \frac{973}{s-49.2}$	0.25
Normalized	$\frac{420}{s+403} \frac{990}{s-50.1}$	0.26
Algorithm 3.7	$\frac{0.1710s+203.88}{s+240.68} \frac{1000}{s-50}$	0.171

It is clear from the above table that the new algorithm out performs all other methods.

3.11.2 Example 3.11: A Fourth Order Unstable System

Consider the following discrete time system:

$$G_n(z) = \frac{2.2256(z-2.0395)(z-0.3460)(z+0.3797)}{(z-1.755)(z-0.8605)(z^2+1.1132\ z+0.3295)}$$

This system was studied in [76] and a model reduction algorithm based on the generalized input normal representation was used. A third order reduced model was obtained with an upper bound on the error given by 0.302.

Algorithm 3.7 was used to obtain the following third order reduced model

$$G_3(z) = \frac{2.2137\ z^3 - 5.6203\ z^2 + 2.4992\ z - 0.5159}{(z-1.755)(z\ -0.8523)(\ z+0.5894)}$$

The true error bound is equal to 0.0834, which is much smaller than the error level reported above. The above algorithm is also used to obtain a strictly proper reduced order model. The reduced order model is given by

$$G_{2,3}(z) = \frac{2.2974\ z^2 - 5.2988\ \ z + 1.9106}{(z-1.6167)(z\ -0.8695)(\ z+0.6321)}$$

and the approximation error is 0.1885. Although this is almost double the approximation error compared to the biproper approximation, it is smaller than the error level reported in [76].

It was observed from the above examples that in most cases, the unstable modes of the original transfer functions are preserved in the reduced order models. However, this is not always the case.

3.12 Frequency Selection Algorithm

In the implementation of the frequency-domain approximation algorithm, it was observed that better performance is obtained with a larger number of samples. However, this is not desirable from computational time point of view. It was observed that the weighting function is becoming zero except at the frequencies at which the error curve have relative maximums. From the discussion in Section 3.11, such frequencies have major effects on the obtained approximation.

Consider the approximation problem in Example 3.1. Figure 3-6, shows a plot of the frequencies at which relative maximum of the error curve occurs. One can observe that the peak frequencies converge to their final values after few iterations. This is a typical behavior in many approximation problems.

Such observation is the basis of the frequency selection scheme presented in this section. The basic approach is divided into two phases. In the first phase we start with equally spaced samples and perform a few iterations of Algorithm 3.7. At the end of these iterations, the regions containing the relative maximums of $U(e^{j\omega_k})$ are obtained and the search is constrained around such frequencies. In the second phase, the regions containing the peaks are reduced further. In implementing this phase, a set of frequencies is selected in each region and the initial value of the weight $U(e^{j\omega_k})$ is taken as that of the frequency of the previous peak. Several iterations are performed before the new region is selected. This process is repeated as required.

An approach that is computationally efficient is presented below. The algorithm is based on Algorithm 3.7 and it is divided into two phases. In the first phase Algorithm 3.7 is run for several iterations with sufficiently large N. In the second phase some extra frequency points are added to the frequencies of interest and some other frequencies are dropped. The decision

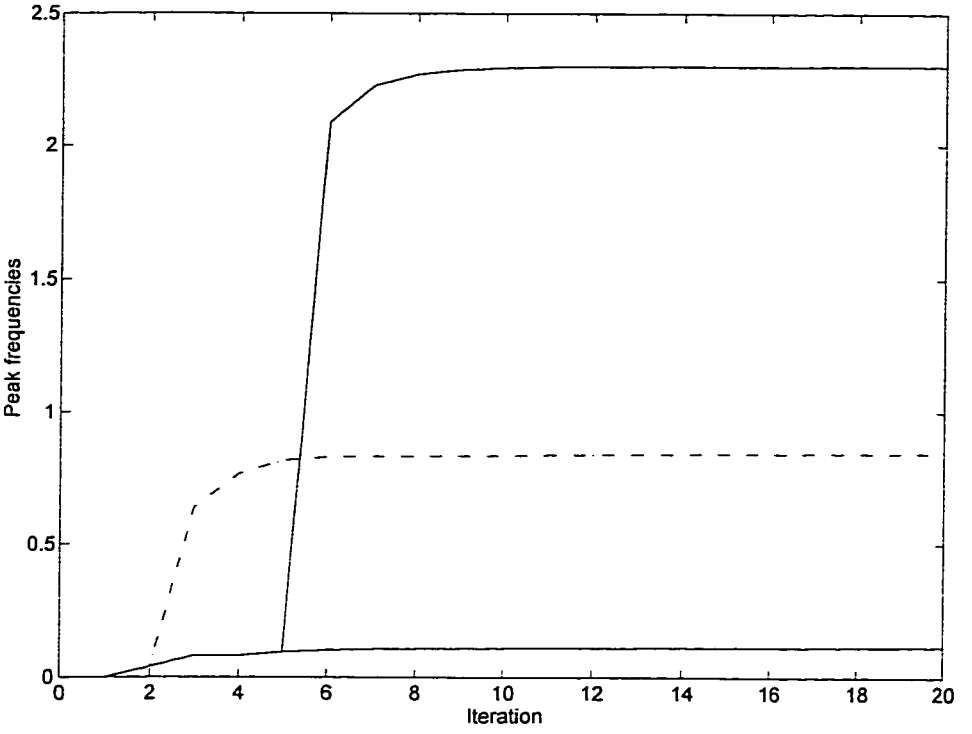

Figure 3-6: Frequencies at which Relative Maximum of the Error Curve Occur

is based on the normalized magnitude of the $U(e^{j\omega_k})$. If $|U(e^{j\omega_k})| < \delta_1$ then ω_k will be removed from Ω. If, on the other hand, $|U(e^{j\omega_k})| > 1 - \delta_2$, then an extra frequency between ω_k and ω_{k+1} is added to Ω. The values of δ_1 and δ_2 are selected from the interval $(0, 1)$. The value of δ_1 is selected to control the rate at which unimportant frequencies are thrown away and δ_2 is selected to determine the regions at which more points are added. The weights $U(e^{j\omega_k})$ for the new frequencies are obtained by linear interpolation of the weights at adjacent frequencies.

3.13 Summary

In this chapter several algorithms to solve the weighted and unweighted H_∞ -norm approximation problems were proposed. Several examples were used to demonstrate the algorithms and compare their performance with other algorithms in current literature.

The algorithms are classified as time-domain algorithms (least squares and output error methods) and frequency-domain algorithms. From solving the many examples we have concluded that the frequency-domain algorithms are better in terms of accuracy and speed than other algorithms. They may obtain the optimal approximation or come very close to it.

The obtained approximations in almost all attempted cases are better than those obtained by other methods and the computational time is reasonable. Modified algorithms based on spectrum smoothing to improve performance are obtained. Attempts to improve the speed of convergence have been made but so far there is no clear strategy to obtain convergence acceleration. The frequency domain algorithms were used to obtain approximation of unstable systems with very good results.

Chapter 4

H_∞ FILTER DESIGN

4.1 Introduction

Digital filters are used to modify, reshape or manipulate signals according to some desired specifications. A large number of algorithms exist for the design of digital filters. It is often desired to minimize the largest amplitude (the H_∞-norm) of the error. The filter design problem can be formulated as an approximation problem and the model reduction algorithms presented in Chapter 3 can be used in the design process. This formulation allows us to solve several filter design problems such as:

- Design FIR and IIR filters with arbitrary (nonclassical) specifications.
- Approximation of an IIR filter by a lower order IIR.
- Approximation of an FIR filter by an IIR filter with or without linear phase.
- Design of IIR filters with no phase constraints.

The filter design problem is formulated in Section 4.2. A new approach is presented in Section 4.3 and several filter design examples are given in Section 4.4.

4.2 The Filter Design Problem

A filter is a frequency selective device that can be specified by its desired frequency response characteristics. The ideal filter can be defined by the desired complex-valued frequency response as follows:

$$H(e^{j\omega}) = H_m(\omega)e^{j\Phi(\omega)} \qquad \text{for } \omega \in [-\pi, \pi]$$

where $H_m(\omega)$ represents the magnitude and $\Phi(\omega)$ represents the phase. The mathematical definition of the H_∞ filter design problem is given below.

Definition 10 *Given an ideal filter characteristic $H(e^{j\omega})$ and a weighting function $W(e^{j\omega})$, find the filter $H_r(z)$ of the following structure*

$$H_r(z) = \frac{n_r(z)}{d_r(z)} = \frac{\sum_{i=0}^{m} b_i z^{m-i}}{z^n + \sum_{i=1}^{n} a_i z^{n-i}}$$

that minimizes

$$\|W(H - H_r)\|_\infty. \tag{4.1}$$

It is usually assumed that $a_i, b_i \in \Re$ and $m \leq n$. Note that the FIR filter

$$H_r(z) = b_0 + b_1 z^{-1} + \ldots + b_n z^{-m}$$

can be obtained as a special case when $d_r(z) = z^m$.

A large volume of literature is available on the design of IIR filters, but there is no analytical solution for obtaining minimax IIR filters. Early minimax designs [79] used the Fletcher-Powell algorithm to solve a simultaneous magnitude and group delay Chebyshev optimization problem. The solution involves complicated nonlinear optimization procedures requiring large amount of computations even for a moderate size problem. Similarly, Cortelazzo and Lightner [80] used the Fletcher-Powell algorithm to design nearly linear phase IIR filters based on a multiple criteria optimization formulation. As stated, their design method requires considerable computing time and is limited to the design of IIR filters with degrees not higher than five. Other approaches leading to suboptimal Chebyshev approximations include linear and nonlinear programming [81, 82], differential correction algorithm [56, 83] and Ellacott-Williams IIR design algorithm [64, 65].

Recently, several authors proposed alternative algorithms for the solution of the minimax IIR filters design. In [84], Lawson's algorithm is used to design IIR digital filters which can approximate arbitrary log magnitude functions. However it was reported in [84] that the resulting filter may not be stable and the algorithm may not converge in some situations. Chen and Parks [66] used linear programming and a modified Ellacott-Williams algorithm to design IIR filters with arbitrary magnitude, phase or group delay. However this method does not guarantee that the best approximation solution can be obtained even in the local optimal sense [66]. A weighted least squares algorithm for magnitude and phase approximation using a modified Lawson's algorithm has been proposed by Lim [85]. Combined features of both algorithms in [66] and [87] are used in [85] to derive a new algorithm with less computational time.

More recently, IIR filter design algorithms based on model reduction were proposed. The general approach is to design FIR filters to satisfy the specifications and then approximate the FIR filter with an IIR filter of the desired order. Algorithms based on balanced truncation [89] and Hankel-norm approximation [88], [90] have been presented. It may be recalled that the Hankel-norm approximation and balanced truncation give suboptimal H_∞-norm approximation. For this reason, it is expected that the filters designed using the algorithms of Chapter 3 will lead to a better result.

4.3 New Filter Design Techniques

The filter design problem can be stated as follows: find the transfer function $G_r(z)$ that best approximates the ideal filter. The H_∞-norm approximation algorithms presented in Chapter 3 can be used to design minimax filters. Identification-based techniques can be used in a two-step procedure. A high order FIR (or IIR) filter is designed to satisfy the specification and then a low order filter is obtained by approximating the high order filter. The frequency-domain algorithm can be used to design the filters directly from specification.

4.3.1 Linear Phase IIR Filters

The linear phase filters are desirable in many applications in order to avoid magnitude distortion in some specific bands. Except for very special cases, such as all pass filters, all IIR filters have a nonlinear phase. Several techniques have been developed to design filters with almost linear phase in the pass band. One such technique is to design an IIR filter that meets the magnitude specification and then design an all-pass equalizer to make the phase linear [79]. Another approach is to design directly the IIR filter that simultaneously meets both magnitude and

phase specifications [80, 86]. A third approach is based on FIR prototyping. This is possible because the FIR filters with linear phase are designed more easily compared to IIR filters. In this approach a linear phase FIR filter that meets the magnitude specifications is designed and then it is approximated by an IIR filter of the desired order.

4.3.2 Magnitude Approximation

There are applications, such as speech processing, where most of the information is carried in the magnitude and the phase does not play any important role. In this situation one can design filters from magnitude specifications only. The phase may assume any arbitrary shape. Relaxing the phase specifications enables the algorithm to meet more efficiently the magnitude specifications. This is evident from the fact that

$$| \, |z_1| - |z_2| \, | \leq |z_1 - z_2|$$

for any complex numbers z_1 and z_2. The following filter design algorithm is a modified version of Algorithm 3.7 which may be used to solve the magnitude approximation problem.

***ALGORITHM* 4.1: Filter Design: Magnitude Approximation**

Given the desired magnitude specification $H(\omega)$, the weighting function $W(\omega)$, the desired order r and the maximum number of iteration l_{max}.

Step 1: Discretize $H(e^{j\omega})$ on N points $\omega_k \in \Omega$, Set $l = 1$, $U^1(e^{j\omega_k}) = 1$ for $\omega_k \in \Omega$.

Step 2: Solve for $G_r = \frac{n_r(e^{j\omega_k})}{d_r(e^{j\omega_k})} = \frac{b_0 + b_1 e^{-j\omega_k} + \ldots + b_r e^{-jr\omega_k}}{1 + a_1 e^{-j\omega_k} + \ldots + a_r e^{-jr\omega_k}}$ from

$$\min_{b_0, b_1, \ldots, b_r, a_1, a_2, \ldots, a_r} \left\| \left(H(e^{j\omega_k}) \, d_r(e^{j\omega_k}) - n_r(e^{j\omega_k}) \right) U^l(e^{j\omega_k}) \right\|_2$$

Step 3: Update $U(e^{j\omega_k})$

$$U^{l+1}(e^{j\omega_k}) = \big|\,|H(e^{j\omega_k})| - |G_r(e^{j\omega_k})|\,\big|\,W(\omega_k)U^l(e^{j\omega_k})/\alpha$$

where α is a scaling factor.

Step 4: If $l = l_{max}$, stop. Otherwise set $l = l + 1$ and go to Step 2.

The main difference between Algorithms 3.7 and the above algorithm is the way the weight is updated. The weight here is modified based on the difference between the magnitudes of the original and the approximate filters, whereas in Algorithm 3.7, the weight is modified based on the magnitude of the difference.

4.4 Examples

4.4.1 Approximation of a High Order IIR Filter by a Low Order One

In filter design, the following situation often occurs : One is given stringent specifications for the design of a filter. Following a standard method, a Chebyshev filter with good equiripple characteristics can be obtained. However, the order can be excessively high. In this situation, it may be desirable to obtain a lower order IIR filter meeting the desired specifications. The low order filter obtained will then closely approximate the Chebyshev filter while meeting the specifications, as shown in the following example.

Example 4.1: Low Pass Filter

It is required to design a low pass filter which has a cutoff frequency of 0.3 and 30 dB ripples in the stop band. A 30^{th} order Chebyshev filter is designed using MATLAB signal processing

Toolbox. The magnitude characteristics are shown in Figure 4-1. Algorithm 3.7 was used to approximate the Chebyshev filter with a 9^{th} order filter. The characteristics of the reduced order filter is also shown in Figure 4-1. The characteristics of the low order filter are marginally inferior to the full order filter. The approximation error was found to be 0.0243.

4.4.2 Approximation of High Order FIR Filter

The design of linear phase FIR filters is a well known problem and many excellent algorithms are available to solve it. In the following example an FIR filter is designed to satisfy the specification and then a low order IIR filter is obtained to approximate the FIR filter.

Example 4.2: Bandpass Filter

Consider the filter having ideal bandpass characteristics shown in Figure 4-2. The optimal linear phase 80^{th} order FIR filter was obtained using the "remez" routine of the MATLAB signal processing Toolbox. The FIR filter is then approximated with an IIR filter of order 23 using Algorithm 3.7. Figure 4-2 shows the frequency response characteristics of both the FIR and IIR filters. The FIR and IIR responses are indistinguishable with error norm $\|E\|_\infty$ equal to 0.0040. The phase of the IIR filter is shown in Figure 4-3. It can be seen that the phase is almost linear in the passband.

4.4.3 Approximation of an Ideal Filter by an IIR Filter with Nearly Linear Phase

Consider the low pass filter specifications shown in Figure 4-4. An eighth order filter has been used to approximate the desired frequency characteristics. The group delay has been selected

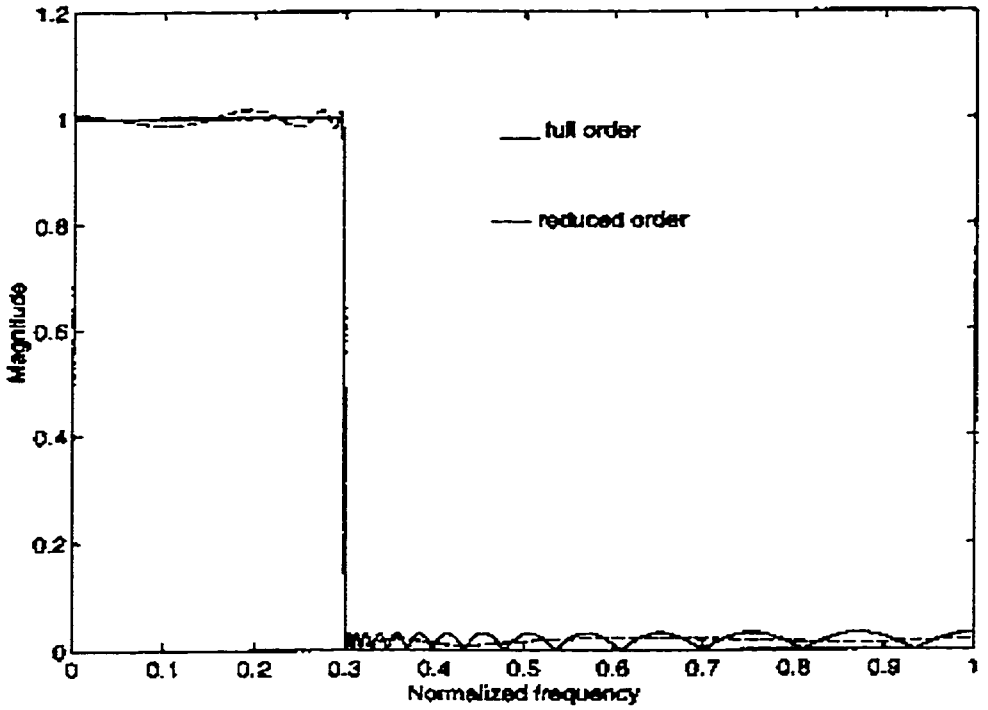

Figure 4-1: Approximation of 30th Order Chebyshev Filter by a 9th Order IIR Filter

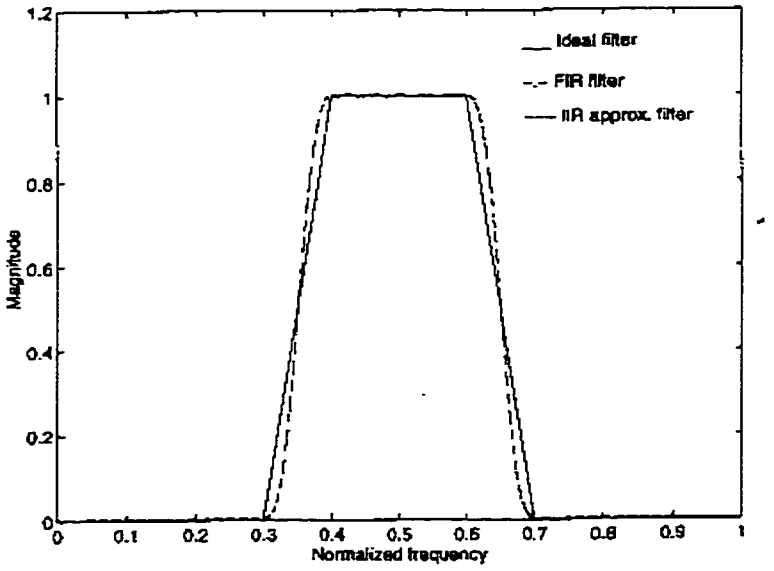

Figure 4-2: Characteristics of the Ideal, FIR and IIR Filters

to be 16 (twice the order of the filter). Figures 4-4 and 4-5 show the magnitude and phase of the 8^{th} order filter. It can be observed that the phase is almost linear in the passband. The approximation error is found to be $\|E\|_\infty = 0.0455$.

4.4.4 Magnitude Approximation Example

Relaxing the phase specifications enables the algorithm to meet more efficiently the magnitude specifications. Algorithm 4.1 can be used to obtain an IIR filter of the desired order to approximate the ideal frequency specifications given in Section 4.4.3. Figure 4-6 shows the magnitude approximation of an 8^{th} order low pass IIR filter. The approximation error is calculated as $\|E\|_\infty = 0.0089$, which is much less than that of Example 4.4.3.

4.4.5 Design of General Shape Filter

Consider the digital filter described by the frequency characteristics in Figure 4-7. It is required to obtain an IIR filter satisfying the specifications. An IIR filter of order 11 was designed using Algorithm 3.7 and the magnitude characteristics of the designed filter are shown in Figure 4-7. If one is concerned with matching the log magnitude, one can modify the way the weights are updated in Algorithms 3.7 and 4.1 so that the weights reflect the error in decibels rather than the absolute value of the error magnitude.

4.5 Summary

The H_∞-norm approximation algorithms developed in Chapter 3 were used in solving several filter design problems. A new algorithm to solve the magnitude approximation problem was proposed to solve filter design problems when only the magnitude characteristics are considered.

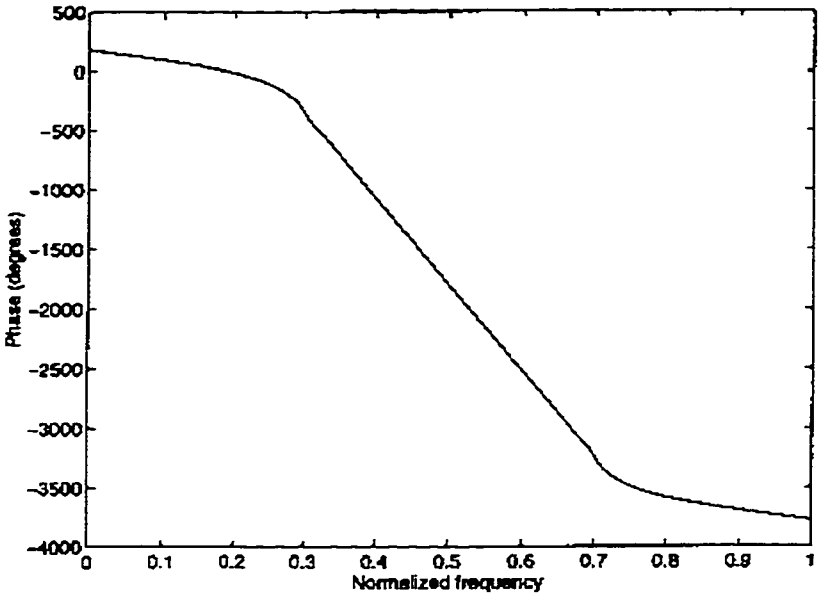

Figure 4-3: Phase Characteristic of The 8 th order IIR Filter

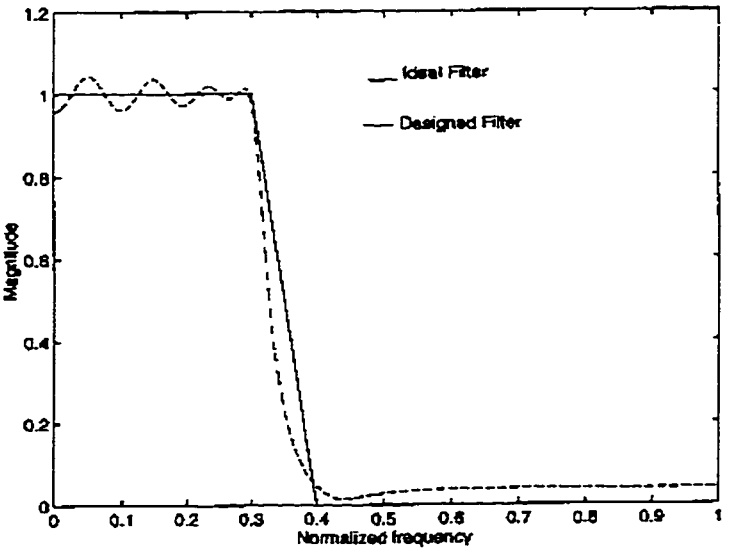

Figure 4-4: IIR Approximation of an Ideal Linear Phase Filter-Magnitude.

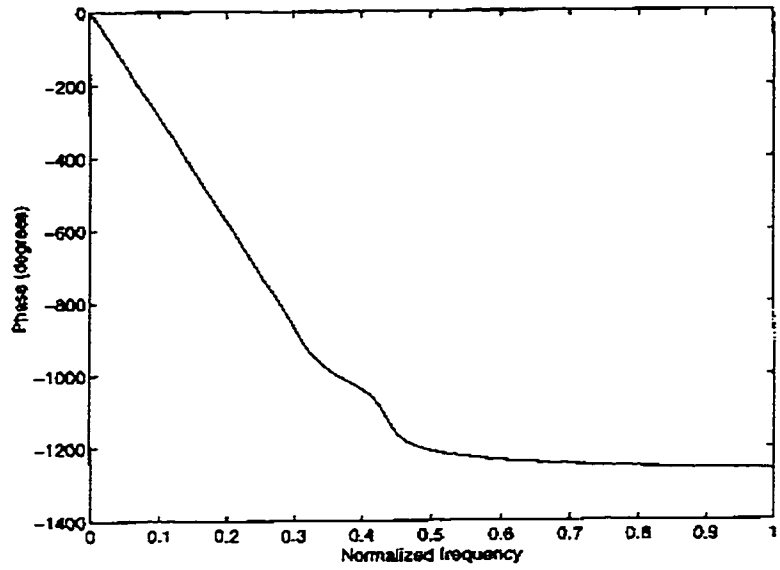

Figure 4-5: IIR Approximation of an Ideal Linear Phase Filter-Phase

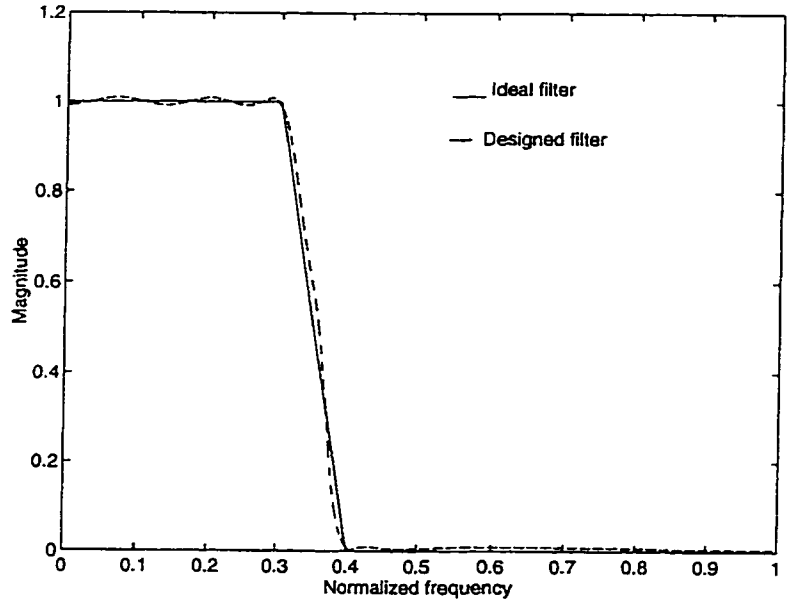

Figure 4-6: Magnitude Approximation-80th Order IIR Filter

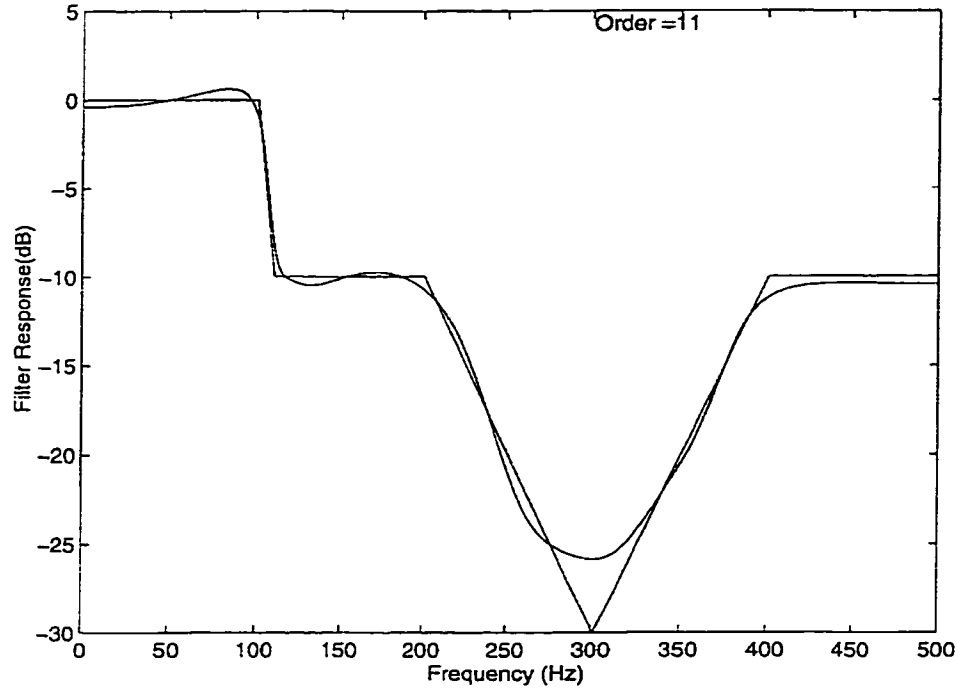

Figure 4-7: Magnitude Approximation of the General Shape Filter in Example 4.4

This scheme allows high flexibility in magnitude, phase and filter order specifications.

Chapter 5

CONTROLLER REDUCTION

5.1 Introduction

Modern controller design schemes may result in high order controllers. It is often desirable to use low order controllers because they are more reliable and are easier to implement. There are three basic approaches that may be used to obtain low order controllers. The direct design approach directly seeks a low order controller for the full order plant [91], [92]. The basic strategy is to come up with a set of necessary conditions on the controller's parameters to achieve optimality. The solution of these conditions is not straightforward and it requires a large volume of computation. In the second approach, the model reduction approach, the plant model is reduced and then a controller is designed for the low order plant model. This controller is then used with the true plant. There are two problems with this approach. First, the resulting controllers may still be of a very high order as in the case when μ-synthesis is used. The second problem is that the approximation is done at an early stage and the effect of approximation error on the behavior of the closed-loop system with the true plant may be difficult to determine. In

the third approach, a controller is designed based on the full order plant model and then model reduction is applied to the controller to obtain a controller of the desired order. This is known as controller reduction. In this chapter, the H_∞-norm model reduction algorithms proposed in Chapter 3 are used to solve several controller reduction problems. The controller reduction problem is discussed in Section 5.2. In Section 5.3, controller reduction for uncertain systems is considered. Reduced order controllers that result in a closed-loop system approximating the original closed-loop system are discussed in Section 5.4. Several controller reduction examples are given in Section 5.5.

5.2 The Controller Reduction Problem

The controller reduction problem is concerned with finding a low order controller that preserves some desirable properties of the original closed-loop system. Applying unweighted model reduction techniques to the controller may not give desirable results in many cases. This can be explained by the fact that errors at some frequencies are critical in preserving satisfactory robustness and performance characteristics, while errors at other frequencies have little or no effect on the closed-loop system. Giving equal weights to all frequencies may limit the possible achievements. Frequency dependent weights are used to emphasize errors at the critical regions. There are several ways which may be used to come up with the weights. However, there is no simple procedure to get the best weights [32, 72].

Enns [32] formulated the controller reduction as a frequency weighted H_∞ model reduction and solved the problem using weighted balanced truncation. Anderson and Liu [72] summarized the main results on controller reduction and compared the performance of the techniques

available at the end of the 1980s.

The controller reduction techniques in literature can be classified into two groups: The additive perturbation form and the coprime factorization approach. In the first approach, the true controller is represented as the sum of the reduced order controller and additive perturbation. Consider the feedback system shown in Figure 5-1. Let K be a stabilizing controller. It is easy to verify that the systems in Figures 5-2 and 5-1 are equivalent. The actual controller consists of the reduced order controller and an additive perturbation.

The main limitation of this approach is that it is restricted to stable controllers. If the controller is unstable, the first step is to factor the controller to a stable and an anti-stable part and apply reduction to the stable part only. This approach has been used in [32], [72], [93] and others. A well known sufficient condition is given in the following lemma.

Lemma 8 *[72] Given a stabilizing controller $K(s)$ for the plant $G(s)$. K_r is a stabilizing controller if*

1. *$K_r(s)$ has the same number of unstable poles as $K(s)$.*

2. *Either*

$$\|(K-K_r)G(I+KG)^{-1}\|_\infty \leq 1, \text{ or} \qquad \|(I+GK)^{-1}G(K-K_r)\|_\infty \leq 1$$

The above lemma suggests the following controller reduction algorithm.

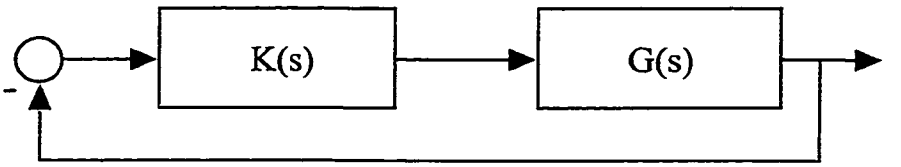

Figure 5-1: Feedback Control System

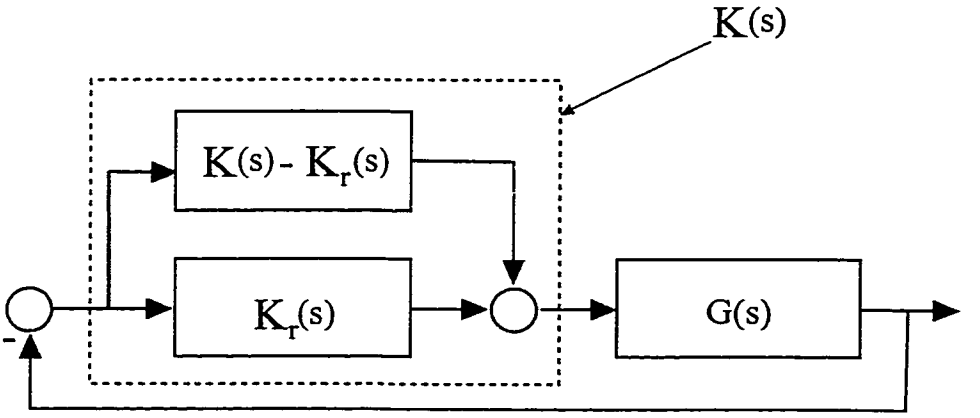

Figure 5-2: Feedback System with Additive Perturbation of the Controller

***ALGORITHM* 5.1: Controller Reduction**

Given the full order plant model G_n and the controller model K_n.

Step 1: Factor $K_n(s)$ as

$$K_n(s) = K_{ns}(s) + K_{nu}(s)$$

where $K_{ns}(s)$ is stable and $K_{nu}(s)$ is anti-stable.

Step 2: Set the weighting function

$$W(s) = G_n(s)(I + K_n(s)G_n(s))^{-1}$$

Step 3: Solve the weighted model reduction

$$\gamma = \min_{K_{rs}} \|W(K_{ns} - K_{rs})\|_\infty$$

Step 4: The reduced order controller is given by

$$K_r = K_{rs} + K_{nu}$$

which is a stabilizing controller if $\gamma \leq 1$.

Note that the above algorithm is useless if the controller is anti-stable. Furthermore, the conditions used to derive the algorithm are only sufficient and other stabilizing controllers that do not satisfy these conditions may exist.

In the coprime factor approach, the plant and controllers are expressed as right (left) coprime factors. Model reduction is applied to these factors to obtain a set of low order right (left)

coprime factors of the controller. This approach has two major advantages: First the approach is directly applicable to both stable and unstable controllers. Furthermore, K and K_r need not have the same number of unstable poles. This approach was first proposed by Liu and Anderson [94] and has been used in [72], [75] and [93].

5.3 Preservation of Stability and Performance

In the previous section, the closed-loop stability is guaranteed with a reduced order controller. However nothing is mentioned about the closed-loop performance. Consider the feedback systems in Figure 5-3 . Let P, K be LTI and let P be partitioned as $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$. The lower linear fractional transformation (LLFT) represents the transfer function between u to y and is defined as:

$$F_l(P(z), K(z)) = P_{11}(z) + P_{12}(z)K(z)(I - P_{22}(z)K(z))^{-1}P_{21}(z)$$

where $(I - P_{22}(z)K(z))$ is non-singular.

Now consider the system in Figure 5-4. The upper linear fractional transformation is the transfer function between u to y and is defined as

$$F_u(P(z), \Delta) = P_{22}(z) + P_{21}(z)\Delta(I - P_{11}(z)\Delta)^{-1}P_{12}(z)$$

where $(I - P_{11}(z)\Delta)$ is non-singular.

In this section reduced order controllers that guarantee some performance specifications are addressed. The problem is formulated in order to obtain a reduced order controller K_r

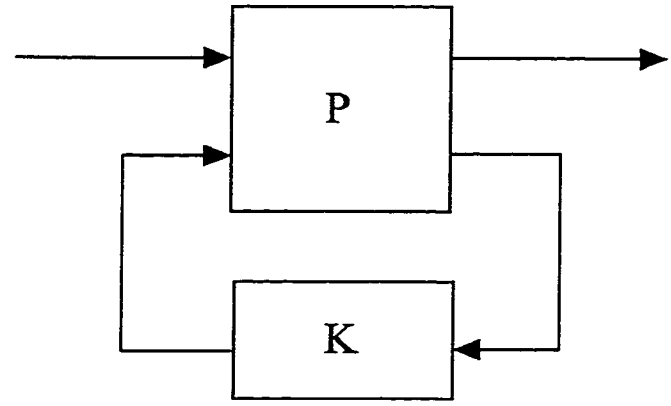

Figure 5-3: Lower Linear Fractional Transformation

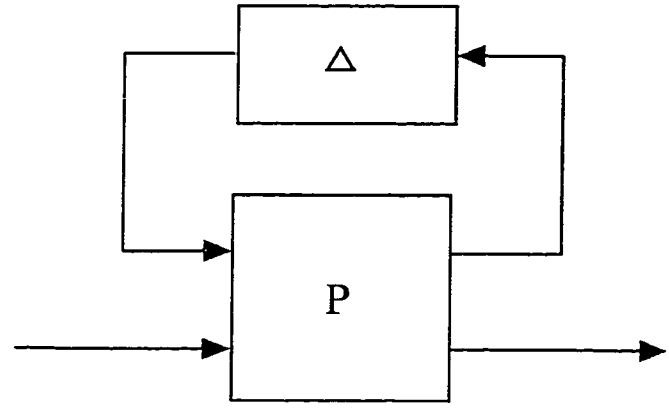

Figure 5-4: Upper Linear Fractional Transformation

such that $\|F_l(P, K_r)\|_\infty < \gamma$. A desirable K_r can be obtained by solving the standard H_∞-suboptimal control problem [73]. The set of all controllers satisfying the performance level are parameterized in terms of a free parameter. The well known central controller is of the same order as the generalized plant. Other controllers are obtained for different selections of the free parameters. Although these controllers have generic orders that are larger than n, there may exist controllers of order less than n that are stabilizing and satisfy the performance level. It is not generally known whether there exist K_r (with $r < n$) that achieves the same performance level and it is not known how to characterize such low order controllers if any do exist. Gu et al [96] proposed a procedure for selecting the free parameter such that K_r has some unobservable modes. Li and Chang [97], have considered a similar approach. Unfortunately, the expected reduction for both methods is small in general.

The problem of reducing controller order while preserving stability and/or performance can be stated as: Find $W_i, W_o \in RH_\infty$ such that

$$\|W_o(K - K_r)W_i\|_\infty < 1 \Leftrightarrow \|F_l(P, K_r)\|_\infty < \gamma.$$

Lenz et al [98] showed that except for special cases it is not possible to find such weighting functions. In [98], an approach to find $W_i, W_o \in RH_\infty$ such that

$$\|W_o(K - K_r)W_i\|_\infty < \frac{1}{\sqrt{2}} \Rightarrow \|F_l(P, K_r)\|_\infty < \sqrt{2}\gamma.$$

Goddard and Glover [99, 93] proposed improved schemes to obtain the weights such that the controller reduction problem will preserve stability and performance. Yang and Packard [100]

proposed a technique based on μ-synthesis.

5.4 Controller Reduction of Uncertain Systems

Uncertain systems are often represented in the standard linear fractional transformation shown in Figure 5-5. The uncertainty is represented by Δ which is, in general, a block diagonal matrix with complex and/or real blocks. This framework covers a large number of problems in robust stability, robust performance and disturbance rejection [73]. A standard tool used in the analysis of robust systems is the structural singular values defined below.

Definition 11 *[73] For $M \in \mathbb{C}^{n\times n}$, the structural singular values of M with respect to uncertainty structure of Δ , denoted by $\mu_{\Delta}(M)$, is defined as*

$$\mu_{\mathbf{\Delta}}(M) = \frac{1}{\{\sigma_{\max}(\Delta) \mid \Delta \in \mathbf{\Delta}, \det(\mathbf{I} - \mathbf{M}\Delta) = 0\}},$$

unless no $\Delta \in \mathbf{\Delta}$ makes $\mathbf{I} - \mathbf{M}\Delta$ singular in which case $\mu_{\mathbf{\Delta}}(M) = 0$.

The generic controller reduction problem can be stated as follows.

Definition 12 ***Generic Controller Reduction Problem:*** *Given $G(s)$ and a robustly stabilizing controller $K(s)$, find an r^{th} order robustly stabilizing controller $K_r(s)$ that has the same number of unstable poles as $K(s)$.*

Note that the condition on $K(s)$ being robustly stabilizing is equivalent to $\mu_{\Delta}(F_l(G,K)) < 1$ and there is no known direct method to obtain K_r that stabilizes the closed-loop system for all possible Δ.

In [101], it was shown that the controller reduction problem for uncertain systems with guaranteed closed-loop performance is equivalent to a weighted H_∞ model reduction problem. An algorithm based on μ-analysis is used to obtain the weights. Consider the block diagrams in Figure 5-8. It is simple to verify that the three block diagram representations are equivalent with $T = F_l(G, K)$. If one defines an uncertainty block Δ such that

$$\sigma_{\max}(\Delta_k) = \sigma_{\max}(K - K_r)$$

then the system in Figure 5-8 is equivalent to the μ-analysis problem in Figure 5-6. Note that some conservatism is introduced here. Solving the sufficient μ-analysis problem, one obtains a frequency dependent upper bound on $\sigma_{\max}(\Delta_k)$. This upper bound can be obtained by performing few bisection steps on the size of Δ_k at each frequency to determine $\sigma_{\max}(\Delta_k(j\omega))$ before instability can occur. With the availability of the upper bound $v(\omega)$, the following sufficient condition for K_r to be robustly stabilizing is that

$$\sigma_{\max}(K - K_r) < v(\omega).$$

Such a controller reduction problem can be cast as the following approximation problem.

$$\min_{K_r} \| W(K - K_r) \|_\infty$$

where $W(s)$ is selected such that

$$|W(jw)| \geq \frac{1}{v(\omega)} \qquad \forall \omega$$

Note that if it is possible to find K_r such that $\|W(K-K_r)\|_\infty < 1$, then the specification is guaranteed. If, on the other hand, $\|W(K-K_r)\|_\infty > 1$, then no conclusion is reached. The controller reduction algorithm is summarized in the following steps.

***ALGORITHM* 5.2: Controller Reduction For Uncertain Systems**

Given $G(s)$, a stabilizing $K(s)$ and a desired order r.

Step 1: Find $T = F_l(G,K)$. At each frequency, perform a μ-analysis procedure to determine an upper bound for $\sigma_{max}(\Delta_k) \leq v(\omega)$, for which the system in Figure 5.7 is robustly stable.

Step 2: Obtain $W(s) = w(s)I$ where $w(s)$ is selected such that

$$|w(j\omega)| \geq \frac{1}{v(\omega)}$$

Step 3: Given r, the desired order, find a $K_r(s)$ with the same number of unstable poles as $K(s)$ by solving the weighted H_∞ model approximation problem

$$\min_{K_r} \ \|W(K-K_r)\|_\infty.$$

Remark 3 *It was shown in [101] that the above algorithm gives a sufficient condition for the controllers to satisfy robust stability.*

Remark 4 *If the weighted H_∞ algorithm is solved by the Algorithm 3.7, then obtaining a rational transfer function $W(s)$ is not needed and one simply needs samples of $\frac{1}{v(\omega)}$ at the desired frequencies.*

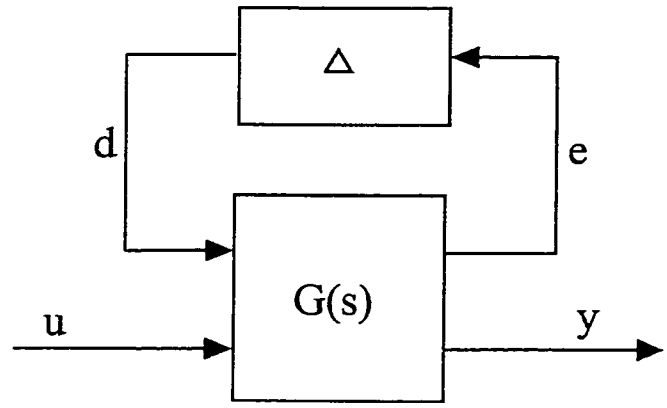

Figure 5-5: A set of Uncertain Plants

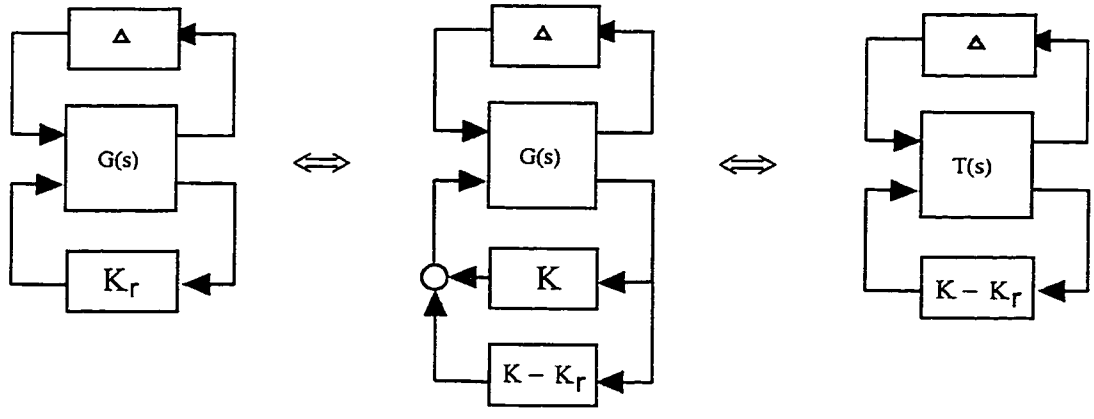

Figure 5-6: Equivalent Blockdiagrams of the Control System

5.5 Transfer Function Approximation

In some applications, it may be desirable to find a reduced order controller K_r such that the closed-loop system with K_r is close to the original closed-loop system. Note that the closed-loop transfer function can be expressed as a linear fractional transformation $F_l(P,K)$ and the transfer function approximation problem can be stated as:

Definition 13 *Given a standard plant $P$and a stabilizing controller K, find an r^{th} order K_r that has the same number of unstable poles as K such that*

$$\|W_1(F_l(P,K) - F_l(P,K_r))W_2\|_\infty \leq \gamma.$$

The transfer function approximation scheme proposed in [91] is not guaranteed to lead to a stabilizing controller. Kavranoğlu [101] showed that this problem is equivalent to a generic controller reduction problem. The transfer function approximation problem can be expressed as shown in Figure 5-5. With simple manipulation, it can be shown that the above problem is equivalent to a generic approximation problem with

$$G(s) = \begin{pmatrix} \frac{1}{\gamma}W_1(s)P_{12}(s)K(s)(I - P_{22}(s)K(s))^{-1}P_{21}(s)W_2(s) & -W_1(s)P_{12}(s) \\ \frac{1}{\gamma}P_{21}(s)W_2(s) & P_{22} \end{pmatrix}$$

and this approach leads to a stabilizing controller that satisfies the norm bound γ.

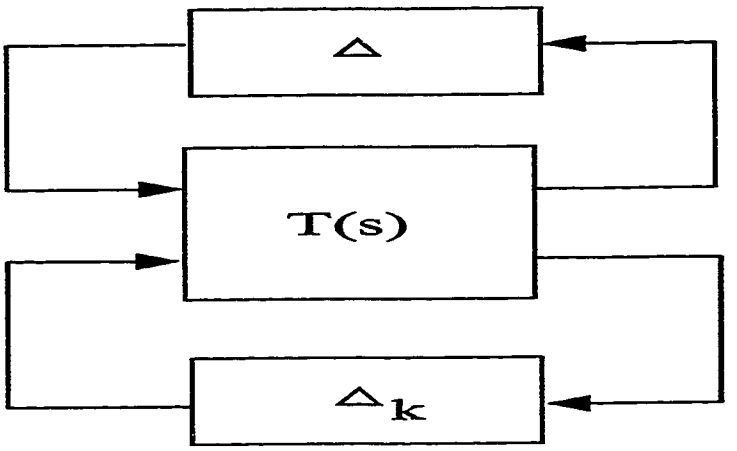

Figure 5-7: Sufficient Mu-Synthesis Problem

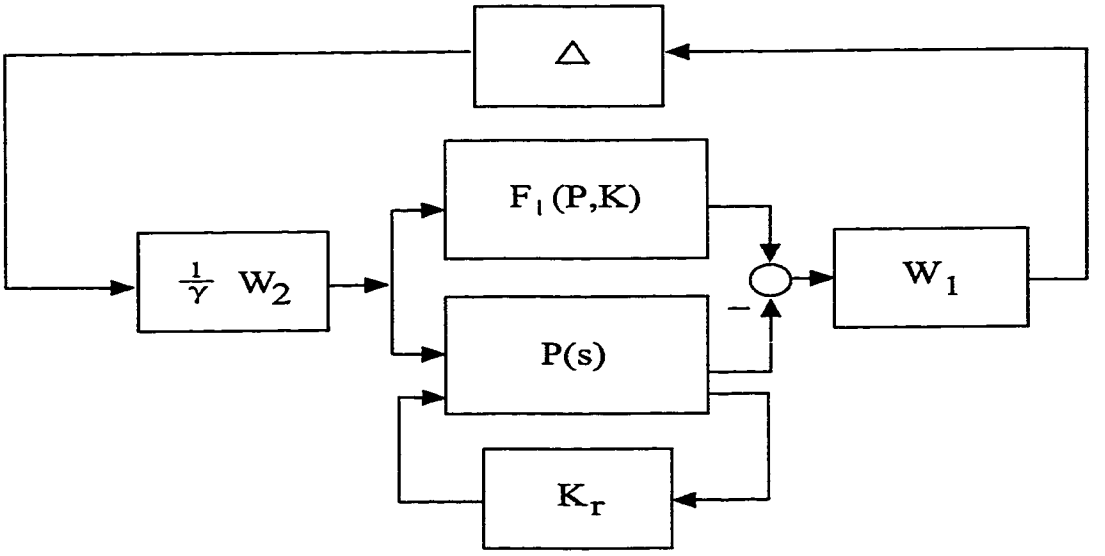

Figure 5-8: Transfer Function Matching Equivalent Blockdiagrams of the Control System

5.6 Examples

In this section several examples are presented to illustrate the above controller reduction algorithms.

5.6.1 Example 5.1

In this example we consider the well known four disk problem [73]. The system is given by

$$\dot{x}(t) = Ax + B_1 w + B_2 u$$

$$z = \begin{bmatrix} H \\ 0 \end{bmatrix} x + \begin{bmatrix} I \\ 0 \end{bmatrix} u$$

$$y = C_2\, x + [0 \;\; I]\; w$$

where

$$A = \begin{bmatrix} -0.161 & -6.004 & -0.58215 & -9.9835 & -0.40727 & -0.3982 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$H = [0\ \ 0\ \ 0\ \ 0\ \ 0.55\ \ 11\ \ 1.32\ \ 18] \times 10^{-3}$$

$$B_1 = [B_2\ \ 0]$$

$$C_2 = [0\ \ 0\ \ 6.4432 \times 10^{-3}\ \ 2.3196 \times 10^{-3}\ \ 7.1252 \times 10^{-2}\ \ 1.0002\ \ 0.10455\ \ 0.99551]$$

We approximate the full order controller by smaller order controllers. The optimal $\|T_{zw}\|_\infty = 1.1272$. An eighth order suboptimal controller is designed using $\gamma = 1.2$. The full order controller is

$$K(s) = \frac{-0.8179s^7 - 0.1574s^6 - 4.9150s^5 - 0.6311s^4 - 8.1819s^3 - 0.5926s^2 - 3.2694s - 0.1057}{s^8 + 1.9376s^7 + 7.6359s^6 + 11.5915s^5 + 18.432s^4 + 19.9626s^3 + 15.746s^2 + 9.103s + 3.822}$$

The weight $W(\omega)$ for the controller reduction problem is obtained, whose plot is given in Figure 5-9. Algorithm 3.7 is then used to solve the resulting weighted H_∞-norm approximation problem.

TABLE 5.1: The Reduced Order Controllers and Their Performance Level

r	$\|W(K-K_r)\|_\infty$	$\|T_{zw}\|_\infty$	K_r
1	12.9408	73.286	$\frac{-0.2785s-.0085}{s+0.3156}$
2	11.2638	1.9681	$\frac{-0.1637s^2-.0975s-.0030}{s^2+0.4426s+0.1109}$
3	1.42145	1.8801	$\frac{-.368s^3-.0188s^2-0.2145s-0.0066}{s^3+.5146s^2+.6570s+0.2436}$
4	0.85247	1.1991	$\frac{-0.1987s^4-0.3072s^3-0.1368s^2-0.1771s-0.0057}{s^4+0.8700s^3+1.0805s^2+0.5920s+0.2062}$
5	0.80708	1.1970	$\frac{-0.3725s^5-0.1908s^4-1.3130s^3-0.1675s^2-0.6386s-0.0201}{s^5+1.5405s^4+3.5694s^3+2.4087s^2 1.9525s+0.7314}$
6	0.33393	1.1971	$\frac{-0.0915s^6-0.5034s^5-0.2927s^4-1.3204s^3-0.1967s^2-0.6004s-0.0195}{s^6+1.1854s^5+3.4540s^4+s^3 3.2926+3.0921s^2+1.7452s+0.7020}$
7	0.23492	1.1957	$\frac{-0.2608s^7-.4390s^6-2.4097s^5-1.3426s^4-4.8059s^3-.8106s^2-2.0312s-0.0649}{s^7+3.0319s^6+7.1582s^5+10.9129s^4+13.4268s^3+10.1391s^2+6.2039s+2.3471}$

From Table 5.1, it is concluded that controllers with order 4 or higher achieve the required

specification, which is the same performance level γ as the full order controller, but we fail to obtain controllers of order three or less which achieve this performance level. When reduced order controllers of lower orders are required we need to relax the specification.

5.6.2 Example 5.2

In the previous example, at least a fourth order controller is needed to satisfy the specifications. Controllers of order less than 4 can be obtained at the price of increasing γ. In this example two approaches for relaxing the performance level γ_1 are considered. In the first approach, the full order controller is designed assuming a performance level $\gamma_1 = 1.2$, but the performance level for reduced order controllers is relaxed to $\gamma_2 = 1.7$. The results are summarized in Table 5.2. In the second approach, the full order controller is designed assuming $\gamma_1 = 1.7$ and thus reduced order controllers are obtained with the same performance level. The results are shown in Table 5.3.

TABLE 5.2: The Reduced Order Controllers for ($\gamma_1 = 1.2, \gamma_2 = 1.7$)

r	$\|W(K-K_r)\|_\infty$	$\|T_{zw}\|_\infty$	K_r
1	2.8797	1.5595	$\frac{-0.1163s-0.0045}{s+0.2606}$
2	2.7385	11.2578	$\frac{-0.0610s^2-0.0146s-0.0001}{s^2+0.1617s+0.0869}$
3	1.0096	1.5017	$\frac{-0.2939s^3-0.0115s^2-0.1769s-0.0031}{s^3+0.2861s^2+0.6738s+0.1575}$
4	0.4087	1.2610	$\frac{-0.1800s^4-0.2551s^3-0.1218s^2-0.1474s-0.0047}{s^4+0.8295s^3+1.0543s^2+0.5464s+0.2015}$

TABLE 5.3: The Reduced Order Controllers for ($\gamma_1 = \gamma_2 = 1.7$)

r	$\|W(K-K_r)\|_\infty$	$\|T_{zw}\|_\infty$	K_r
1	0.7967	1.5541	$\frac{-0.0907s-0.0026}{s+0.1722}$
2	0.7117	1.5199	$\frac{-0.0600s^2-0.0390s-0.0014}{s^2+0.3611s+0.0835}$
3	0.3406	1.5228	$\frac{-0.0933s^3-0.0084s^2-0.0544s-0.0017}{s^3+0.3556s^2+0.4843s+0.1080}$
4	0.1576	1.5241	$\frac{-0.0272s^4-0.0977s^3-0.0234s^2-0.0573s-0.0022}{s^4+0.6308s^3+0.9261s^2+0.3916s+0.1291}$

It was concluded from these examples and others that if one can relax the performance level of γ, it is better to relax it at the stage of designing the full order controller and then apply the reduction technique.

5.6.3 Example 5.3

In this example we apply the transfer function approximation to the four disk problem described in Section 5.6.1. The reduced order controller is obtained by minimizing the weighted H_∞-norm of transfer function mismatch. In this example, a suboptimal H_∞ controller with performance level $\gamma_1 = 1.2$ is designed and the transfer function approximation scheme in Section 5.5 is used to obtain the reduced order controllers. $W(\omega)$ is selected such that it is equal to unity in the frequency range of interest and zero elsewhere. The acceptable error level in transfer function matching is taken as 0.15.

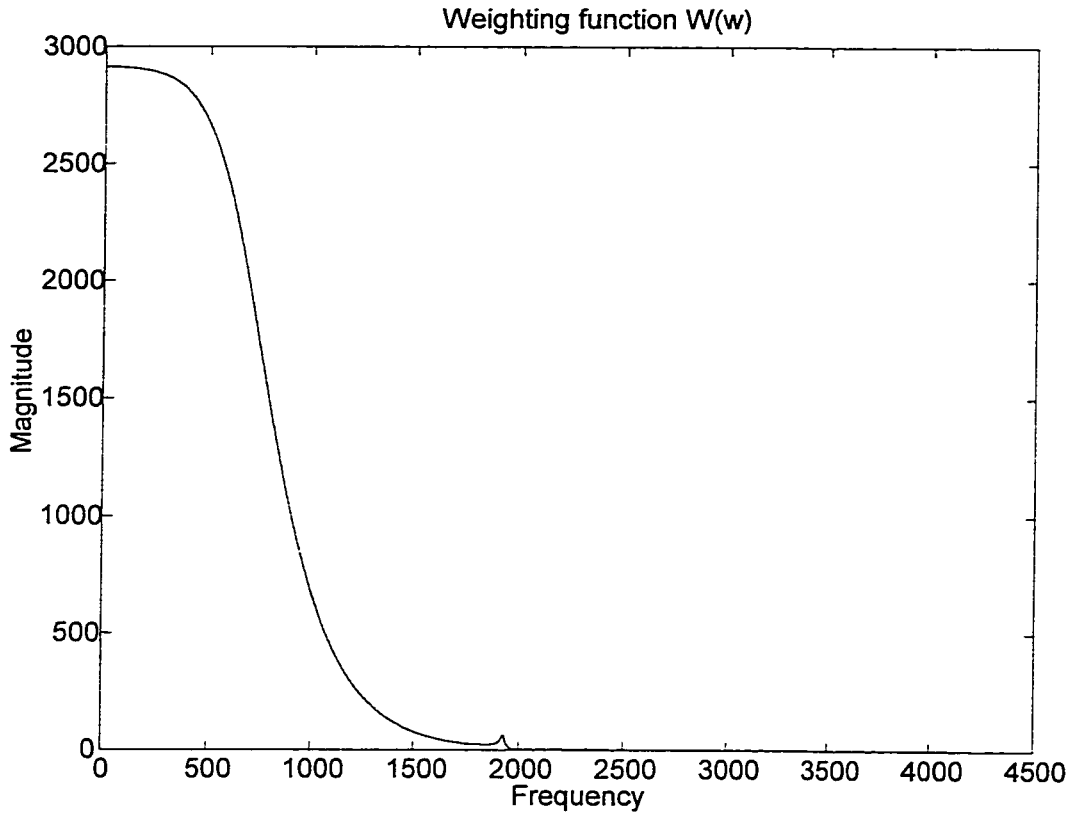

Figure 5-9: The Weighting Function for the Controller Reduction Example

TABLE 5.4: The Reduced Order Controllers for ($\gamma_1 = 1.2$)

r	$\|W(K - K_r)\|_\infty$	$\|T_{zw}\|_\infty$
1	6.5673	2.6995
2	7.2282	2.6323
3	1.9415	1.3745
4	1.1894	1.2254
5	1.5656	1.2725
6	0.4235	1.1969
7	0.3043	1.1962

From the second column of Table 5.4, the weighted error is less than 1 for $r \geq 6$ which is a condition sufficient to guarantee the desired specifications.

5.7 Summary

The controller reduction problem can be posed as a weighted H_∞ -norm approximation problem. In this chapter, we implemented a controller reduction algorithm for uncertain systems. The weights are obtained by solving a μ-synthesis problem. Algorithm 3.9 is then used to solve the resulting weighted approximation problem directly to obtain the reduced order controller. Several well-known examples are used to illustrate the controller reduction scheme.

Chapter 6

IDENTIFICATION IN H_∞

6.1 Introduction

The main objective of identification is to come up with a nominal model describing the behavior of the system under consideration. Identification in H_∞ is concerned with obtaining algorithms that map a set of measurements into a nominal model and an ∞-norm bound on the worst case error. Such a bound is needed to come up with conclusions on robust stability, robust performance or other H_∞-related criteria. In this chapter a new algorithm for H_∞-identification is proposed. The main part of the algorithm is a modified version of Algorithm 3.7. The identification in H_∞ problem is formulated and the related literature is presented in Section 6.2. The new scheme is presented in Section 6.3. A general identification algorithm is proposed in Section 6.4, and several examples which serve to illustrate the new algorithms are given in Section 6.5.

6.2 Identification in H_∞

The problem of H_∞ identification was first formulated in 1989 by Helmicki, Jacobson and Nett [102]. Roughly speaking, the problem is to find an algorithm that maps the given experimental data into a nominal model and an ∞-norm bound on the worst case identification error. In [102], an algorithm based on polynomial interpolation was proposed. Since then, many other algorithms have appeared. Among them: spline interpolation algorithms [103], least squares interpolation algorithms [104], Jackson polynomial technique, Cesaro-sum based algorithm [105], Pick's interpolation algorithm [106] and Fejer interpolation [107]. The problem was extended to the continuous-time case and several algorithms were proposed (See, for example, [108], [109]).

The algorithms can be classified as linear or nonlinear, tuned or untuned. An algorithm is linear if all the operations performed on the data to come up with the nominal model are linear. An algorithm is tuned if the a priori information is used explicitly to come up with the nominal model. If the worst case error approaches zero as the number of observations goes to infinity and the noise level goes to zero, then the algorithm is said to be robustly convergent. Partington [110] has shown that no linear untuned interpolative algorithm can be robustly convergent.

Time-domain algorithms have been presented in [110-112]. However, the problem is more difficult. In [113], the algorithm assumes the availability of the denominator polynomial in addition to experimental data to come up with the numerator and the error bound.

There are several drawbacks with most existing H_∞-identification techniques. Identification in H_∞ schemes use techniques that lead to bounds on the approximation error. With such restrictions, the obtained nominal models may not be good representatives of the true system. Another problem is that the measured frequency response will not affect the bounds for most

algorithms and can be determined before the identification experiment is actually performed. Most of the algorithms reported in the literature provide nominal models in terms of impulse response coefficients and if a rational transfer function is needed, extra errors may be introduced.

The algorithm proposed in this chapter does not suffer from these problems. The nominal model is obtained as the best approximation, in the ∞-norm sense, to the system described by the given frequency response samples. The proposed algorithm gives rational transfer function of the desired order directly.

In this chapter, we consider linear discrete-time shift invariant single-input single-output stable systems. Such systems can be described by the following representation:

$$y(t) = g(t) * u(t)$$

where u, y and g denote the input, the output and the impulse response respectively. In frequency domain, the transfer function is defined as

$$G(z) = \sum_{k=0}^{\infty} g(k)\, z^k. \tag{6.1}$$

Note that this definition is not the standard definition used in the control literature, but it is quite common in the mathematical literature. In the standard definition, the system is stable if all the poles are inside the unit disk. In the notation of (6.1), the system is stable if all the poles are outside the unit disk. This notation will be used in Sections 6.2 and 6.3 to derive the ∞-norm bound of the error. In the rest of the dissertation the usual definition will be used. The proposed scheme uses the standard assumptions in [102]. It is assumed that a set of

N uniformly spaced frequency response measurements is available and there exist two known constants $\rho > 1, M > 0$ such that

$$\sum_{k=0}^{\infty} |g(k)|\rho^k < M. \tag{6.2}$$

Systems satisfying (6.2) are denoted by $H_s(M, \rho)$. Note that ρ is related to the rate of decade of the impulse response coefficients. All the poles of the system should be outside the circle of radius ρ centered at the origin. The value of M can be taken as an upper bound on the worst case steady state gain of the system over all exponentially weighted sinusoidal input. The condition $G(z) \in H_s(M, \rho)$ implies that

$$\sup_{|z| \leqslant \rho} |G(z)| < M.$$

The frequency response is measured at the discrete frequencies

$$\omega_k = \frac{2\pi(k-1)}{N} \qquad\qquad k = 1, 2, \cdots, N$$

and the measured response is assumed to be expressed as

$$E_N^k = G(e^{j\omega_k}) + \epsilon_k \qquad\qquad k = 1, 2, \cdots, N \tag{6.3}$$

where $G(.)$ is the true transfer function and ϵ_k is a bounded noise

$$|\epsilon_k| \leq \varepsilon.$$

The symbol ε is used to denote the noise level. The assumption of bounded noise in (6.3) is

essential in H_∞-identification, and in many cases this assumption is justifiable.

The problem to be solved can be stated as follows: Given the real constants M, ρ, ε and a set of N noisy frequency response measurements $\{E_N^1, E_N^2, \cdots, E_N^k\}$ that comes from an unknown system $G(z)$ which belongs to the set $H_s(M, \rho)$ with noise level ε. It is desirable to find an r^{th} order model $G_r(z)$ and an upper bound on the worst case error $e_N(M, \rho, \epsilon)$ such that

$$\|G - G_r\|_\infty \leq e_N(M, \rho, \epsilon).$$

6.3 The Proposed Algorithm

In this section, we extend the Algorithm 3.7 to obtain the nominal model. The nominal model is obtained by solving an H_∞-norm approximation problem. The only modification here is that the noisy frequency response measurements are used instead of the true values which are unknown in this case.

Assume the availability of N equally spaced frequency response measurements $\{E_N^k, k \in [1, N]\}$. The nominal model is obtained using the following algorithm.

***ALGORITHM* 6.1: H_∞-Identification**

Given $\{E_N^1, E_N^2, \cdots, E_N^N\}$ on the frequencies $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$, the number of iterations l_{max}, and the desired order r.

Step 1: Let $l = 1$, $U^l(e^{j\omega_k}) = 1$ for $k \in [1, N]$

Step 2: Solve for $G_r^l = \frac{n_r^l(e^{j\omega_k})}{d_r^l(e^{j\omega_k})} = \frac{b_0 + b_1 e^{-j\omega_k} + b_2 e^{-j2\omega_k} + \ldots + b_r e^{-jr\omega_k}}{1 + a_1 e^{-j\omega_k} + a_2 e^{-j2\omega_k} + \ldots + a_r e^{-jr\omega_k}}$ from

$$\min_{a_1, a_2, \ldots, a_r, b_0, b_2, \ldots, b_r} \left\| (E_N^k \, d_r^l(e^{j\omega_k}) - n_r^l(e^{j\omega_k})) \, U^l(e^{j\omega_k}) \right\|_2$$

Step 3: The model fitting error at the l^{th} iteration is given by

$$\varepsilon_l = \max_k \left| E_N^k - G_r^l(e^{j\omega_k}) \right|$$

Step 4: Update the weight using

$$U^{l+1}(e^{j\omega_k}) = \left(E_N^k - G_r^l(e^{j\omega_k})\right) U^l(e^{j\omega_k})$$

and scale U^{l+1} as

$$U^{l+1}(e^{j\omega_k}) = \frac{U^{l+1}(e^{j\omega_k})}{\max_k |U^{l+1}(e^{j\omega_k})|}$$

Step 5: If $l = l_{max}$, stop. Otherwise set $l = l + 1$ and go to Step 2.

As discussed in Chapter 3, the model fitting error ε_ℓ is not monotonic in general, so one needs to keep a record of the smallest ε_ℓ and the corresponding transfer function. Let the smallest achievable error be denoted by ε_{min}

$$\varepsilon_{min} = \min_{l \in [1, l_{max}]} \varepsilon_l.$$

It is expected that ε_{min} decreases as the model order increases. If the true system is of finite dimension, then no significant reduction in the error is expected when the order increases beyond a certain level.

6.3.1 Error Bound

In this section we derive an upper bound on the additive error between the identified model and the true model. The true model is not known, but it is assumed to belong to the set $H_s(M,\rho)$. We start by studying the effect of the number of frequency response samples, N, on the error. Recall that the ∞-norm is defined as

$$\|G\|_\infty = \sup_{\omega\in[0,2\pi]} \left|G(e^{j\omega})\right|. \tag{6.4}$$

A direct approach to compute the ∞-norm of a given transfer function $G(z)$ is

$$\|G\|_\infty \approx \max_{k\in[1,N]} \left|G(e^{j\frac{2\pi k}{N}})\right| \tag{6.5}$$

where N is a sufficiently large number. In the following we will investigate the accuracy of the estimate of the ∞-norm obtained using (6.5). The following important lemma from complex variable theory will be used in deriving the bound.

Lemma 9 *(Ahlfors [114], page 179)*

Let $G(z)$ be analytic in a region Γ containing z_0. It is possible to express $G(z)$ in the following Taylor series

$$G(z) = G(z_0) + G^{(1)}(z_0)(z-z_0) + \cdots + \frac{G^{(m)}(z_0)}{m!}(z-z_0)^m + R_{m+1}(z,z_0)$$

where $G^{(m)}$ *represents the* m^{th} *derivative and* R_{m+1} *represents the remainder. A bound on the remainder is given by*

$$|R_{m+1}(z, z_0)| \leq \frac{M_1 |z - z_0|^{m+1}}{\beta^m (\beta - |z - z_0|)}$$

where

$$M_1 = \sup_{z \in D_\beta} |G(z)|, \quad D_\beta = \{z : |z - z_0| = \beta\}$$

and $|z - z_0| \leq \beta$ *is inside* Γ.

The following theorem gives a bound on the maximum approximation error if (6.5) is used instead of (6.4) when $G(z) \in H_s(M, \rho)$.

Theorem 1 *Given* $M > 0$, $\rho > 1$, *and* $N > \frac{\pi}{\rho - 1}$. *If* $G(z) \in H_s(M, \rho)$ *then*

$$\|G\|_\infty - \max_{k \in [1,N]} \left| G\left(e^{j\frac{2\pi k}{N}}\right) \right| \leq \frac{M\pi}{N\rho - \pi}$$

Proof.

Let $\Gamma = \{z : \quad |z| \leqslant \rho\}$; $G(z) \in H_s(M, \rho)$ implies that $G(z)$ is analytic on Γ and

$$\sup_{z \in \Gamma} |G(z)| < M.$$

Define N disks centered on $e^{j\frac{2\pi k}{N}}$ with $k \in [1, N]$ and each has a radius equal to $\frac{\pi}{N}$. It is clear that the N disks are entirely inside Γ. The complex function $G(z)$ inside the k^{th} disk can be expressed as

$$G(z) = G\left(e^{j\frac{2\pi k}{N}}\right) + R_1\left(z, e^{j\frac{2\pi k}{N}}\right)$$

Applying Lemma 9 with $m = 0$ and $\beta = \rho$, one gets

$$\begin{aligned} \left|G(z) - G(e^{j\frac{2\pi k}{N}})\right| &\leq \frac{M\,|z - e^{j\frac{2\pi k}{N}}|}{\rho - |z - e^{j\frac{2\pi k}{N}}|} \\ &\leq \frac{M\,\frac{\pi}{N}}{\rho - \frac{\pi}{N}} = \frac{M\pi}{N\rho - \pi}\,. \end{aligned}$$

Using the fact that for any complex numbers z_1 and z_2

$$|\,|z_1| - |z_2|\,| \leq |\,z_1 - z_2|$$

then for all z inside the circle centered on $e^{j\omega_k}$ with radius $\frac{\pi}{N}$, the following holds

$$\left|\,|G(z)| - \left|G(e^{j\frac{2\pi k}{N}})\right|\,\right| \leq \frac{M\pi}{N\rho - \pi}\,.$$

Since this is true for all $k \in [1, N]$, and the unit circle is contained in the union of the small N disks, then

$$\left|\,\|G\|_\infty - \max_{k\in[1,N]} \left|G(e^{j\frac{2\pi k}{N}})\right|\,\right| \leq \frac{M\pi}{N\rho - \pi} \qquad \blacksquare$$

Now we are ready to present the worst case error bound on the identified nominal model.

Theorem 2 *Given $M > 0$, $\rho > 1$, $N > \frac{\pi}{\rho - 1}$ and assuming that $G(z) \in H_s(M, \rho)$, then the worst case error between the true model and the nominal model produced by Algorithm 6.1 is*

given by

$$e_N(M,\rho,\epsilon) = \epsilon + \varepsilon_{min} + \frac{M\pi}{N\rho - \pi}$$

where ϵ is the measurement noise level and ε_{min} is the minimum achievable error in Algorithm 6.1.

Proof. Define G_N and G_ϵ as follows

$$\begin{aligned} G_N(e^{j\omega}) &= G\left(e^{j\frac{2\pi(k-1)}{N}}\right) \quad \text{for } \omega \in \left[\frac{2\pi(k-1)}{N} - \frac{\pi}{N}, \frac{2\pi(k-1)}{N} + \frac{\pi}{N}\right) \text{ and } k \in [1,N] \\ G_\epsilon(e^{j\omega}) &= E_N^k \qquad\qquad \text{for } \omega \in \left[\frac{2\pi(k-1)}{N} - \frac{\pi}{N}, \frac{2\pi(k-1)}{N} + \frac{\pi}{N}\right) \text{ and } k \in [1,N] \end{aligned}$$

We then have

$$\begin{aligned} \|G_N - G_\epsilon\|_\infty &= \sup_{\omega\in[0,2\pi]} |G_N(e^{jw}) - G_\epsilon(e^{j\omega})| \\ &= \max_{k\in[1,N]} \left|G\left(e^{j\frac{2\pi(k-1)}{N}}\right) - E_N^k\right| \leq \epsilon\,. \end{aligned}$$

Algorithm 6.1 is used to obtain $\hat{G}$ and ε_{min} such that

$$\|G_\epsilon - \hat{G}\|_\infty \leq \varepsilon_{min}$$

and Theorem 1 implies

$$\|G - G_N\|_\infty \leq \frac{M\pi}{N\rho - \pi}.$$

The worst case error $e_N(M,\rho,\varepsilon) = \|\,G - \hat{G}\,\|_\infty$ and the upper bound is derived as follows:

$$
\begin{aligned}
\|G-\hat{G}\|_\infty &= \left\|G-G_N+G_N-G_\epsilon+G_\epsilon-\hat{G}\right\|_\infty \\
&\leq \|G-G_N\|_\infty+\|G_N-G_\epsilon\|_\infty+\|G_\epsilon-\hat{G}\|_\infty \\
&\leq \frac{M\pi}{N\rho-\pi}+\epsilon+\varepsilon_{min}. \qquad \blacksquare
\end{aligned}
$$

The above theorem shows that the worst case error consists of three terms: the measurement noise level, the model fitting error estimated in Algorithm 6.1 and a term due to the use of a finite number of samples and the a priori information. The selection of the model order will only affect ε_{min}. The sum of the other terms gives a lower bound on the least achievable worst case error for all order.

6.4 General H_∞-Identification Problem

In this section we consider the following generalization of the identification problem in Section 6.3. The problem is stated as follows: Given a set of frequency response measurements, it is desired to find a nominal model G_r of order r and a frequency weighting function $W(z)$ of order r_w such that the actual transfer function is in the set

$$G(z) = G_r(z) + W(z)\Delta(z)$$

with $\|\Delta(z)\|_\infty = 1$ and $W(z)$ satisfying the following assumptions

1. $W(z)$ is a rational, stable, minimum phase transfer function of order r_w,

2. $W(z)$ is as tight as possible.

The problem in section 6.3 is a special case with $W(z)$ being a scalar independent of the frequency. To solve this problem the following approach is proposed.

***ALGORITHM* 6.2: General H_∞-Identification**

Given $\{E_N^1,\ E_N^2, \cdots, E_N^N\ \}$ on the frequencies $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$, the number of iterations l_{max}, $M > 0$, $\rho > 1$, $N > \frac{\pi}{\rho - 1}$ and the desired orders r and r_w.

Step 1: Obtain $G_r(z)$ using Algorithm 6.1.

Step 2: Compute the model fitting error

$$\mathcal{E}_N^k = \left| E_N^k - G_r^l(e^{j\omega_k}) \right| + \frac{M\pi}{N\rho - \pi} + \epsilon$$

Step 3: Obtain $W(z)$ of order r_w such that

$$\left|W\left(e^{j\omega_k}\right)\right| \geq \mathcal{E}_N^k \ \text{ and } \ \left|W\left(e^{j\omega_k}\right)\right| - \mathcal{E}_N^k \ \text{ is as small as possible for all } k \in [1, N].$$

The problem in Step 3 can be formulated as a constrained optimization problem. An alternative way is to use a bounding algorithm similar to that in [115] to obtain $W(z)$.

6.5 Examples

In this section four examples are given to illustrate the proposed algorithms.

6.5.1 Example 6.1

Assume that the true model is given by

$$G(z) = \frac{0.5\,z^2 + 0.6\,z + 0.3}{z^2 + 0.4\,z + 0.2}$$

It can be verified that $G(z) \in H_s(2.8, 1.9)$. Let the noise be generated by

$$\varepsilon_k = \epsilon\, e^{j\theta}$$

where the measurement noise level $\epsilon = 0.1$ and $\theta \in [0, 2\pi]$ is a uniformly distributed random variable. N is selected to be 512. The Algorithm 6.1 uses the noisy measurements to obtain the nominal models of orders $1, 2$ and 3. The result is given in Table 6.1.

TABLE 6.1: Identified Models and Error Bounds for Example 6.1.

r	$\hat{G}_r$	$e_N(M, \rho, \epsilon)$	$\|G - \hat{G}_r\|_\infty$
1	$\frac{0.4885z+0.3469}{z-0.1052}$	0.3308	0.1228
2	$\frac{0.5000z^2+0.6000z+0.3000}{z^2+0.4000z+0.2000}$	0.2091	3.7129×10^{-13}
3	$\frac{0.5000z^3+0.3989z^2+0.0587z-0.1207}{z^3-0.0022z^2+0.0391z-0.0804}$	0.2091	7.9704×10^{-13}

The error bound is almost equal for the cases $r = 2$ and $r = 3$. This suggests that $r = 2$ is a suitable choice of the model order. The third column in the table shows the error between the true and the identified models.

6.5.2 Example 6.2

In this example we use the seventh order system in [106] as the true system. The true system is assumed to be

$$G(z) = \frac{0.0420z^6 + 0.2674z^5 + 0.2736z^4 + 0.1691z^3 + 0.5229z^2 + 0.2618z + 0.1410}{z^7 + 0.3557z^6 + 0.2973z^5 + 0.4575z^4 + 0.3671z^3 + 0.0776z^2 + 0.1111z + 0.2897}$$

It can be verified that $G(z) \in H_s(2.5045, 1.1)$. A set of noisy measurements is generated by introducing additive noise of the form $\varepsilon_k = 0.1\, e^{j\theta}$ with $\theta \in [0, 2\pi]$ the uniformly distributed random variable. The Algorithm 6.1 uses the noisy measurements to obtain rational nominal models of different orders. The error bounds obtained using Theorem 6.3 and the true errors are given in Table 6.2 for $N = 32, 512$ and 1024.

TABLE 6.2: Worst-Case Error Bound and the True Error for Example 6.2.

Order r	1	2	3	4	5	6	7
$e_{32}(2.5045, 1.1, .1)$	1.1159	0.9746	0.9574	0.9158	0.6253	0.5626	0.4155
True Error	0.7911	0.6327	0.6512	0.6291	0.2651	0.2066	0.0702
$e_{512}(2.5045, 1.1, .1)$	0.9226	0.8085	0.8145	0.7362	0.3959	0.3632	0.2154
True Error	0.7672	0.6192	0.6145	0.5445	0.2242	0.1772	0.0280
$e_{1024}(2.5045, 1.1, .1)$	0.9120	0.7797	0.7651	0.7301	0.3779	0.3548	0.2088
True Error	0.7431	0.6011	0.5865	0.5513	0.1836	0.1609	0.0069

From Table 6.2, it is clear that as N increases the identified model is getting closer to the true model and bound is reduced. The error bound decreases as the order increases. The effect of the order on the error bound was investigated for the case $N = 512$ and $1 \leq r \leq 10$. The

result is shown in Figure 6-1. From Figure 6-1, it is clear that increasing the order beyond 7 will not significantly change the error bound and therefore the order should be selected to be equal to seven or less.

6.5.3 Example 6.3

Given a set of 512 equally spaced frequency response samples. They are generated from the true system used in Example 6.1 with the noise level $\epsilon = 0.4$. Figure 6.2 shows the magnitude plot of the experimental data together with the true system. Algorithm 6.1 was used and the following second order model was obtained.

$$\hat{G}_2 = \frac{0.4967z^2 + 0.5909z + 0.2936}{z^2 + 0.3962z + 0.2058}$$

The worst case error bound is $e_N(M, \rho, \epsilon) = 0.8069$ and the actual error is $\|G - \hat{G}_r\|_\infty = 0.0128$. Despite the high level noise, the model obtained is very close to the true model.

6.5.4 Example 6.4

Given a set of 512 equally-spaced frequency response measurements. Assume the data comes from the true system used in Example 6.2 with noise level $\epsilon = 0.1$. It is assumed that $G(z) \in H_s(2.5045, 1.1)$, and it is desired to fit the data using the model

$$G(z) = G_r(z) + W(z)\Delta$$

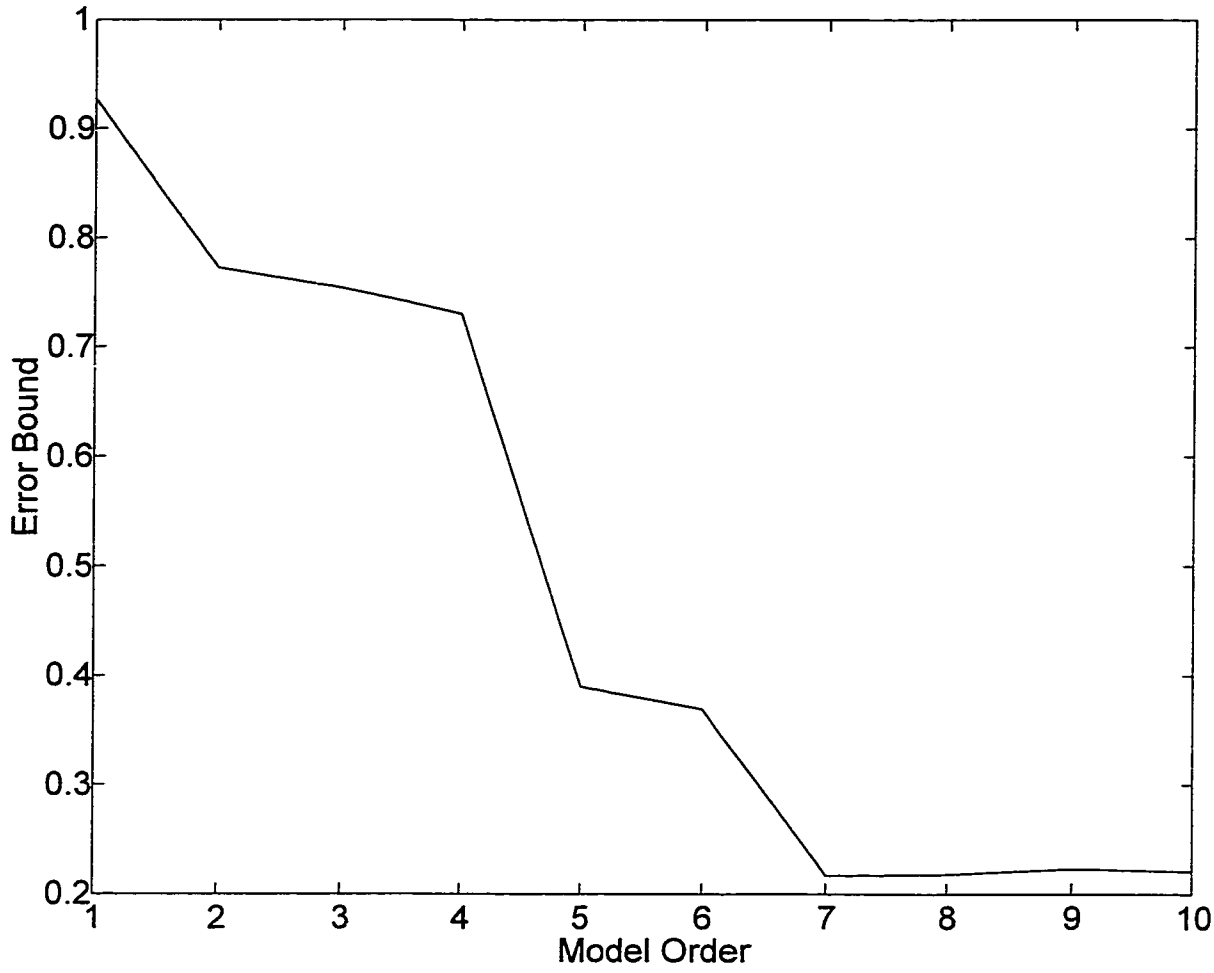

Figure 6-1: Effect of Model Order on Worst-Case Error-Example 6.2

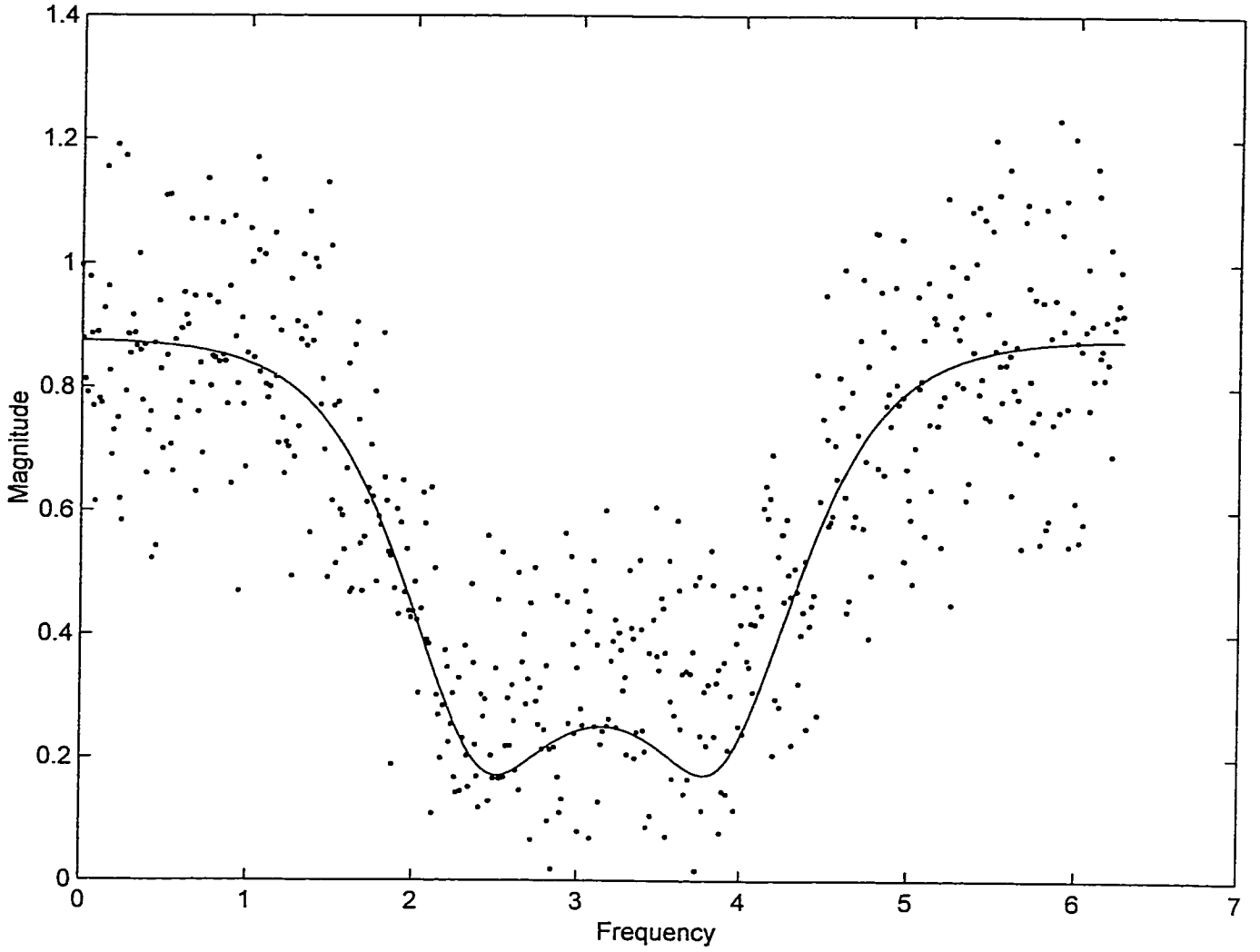

Figure 6-2: The True and The Experimental Frequency Response

where $r = 3$ and $r_w \leq 2$. Algorithm 6.2 is used and the result obtained is given below.

$$G_r(z) = \frac{0.0443z^3 + 0.1398z^2 + 0.0360z + 0.0904}{z^3 - 0.3274z^2 + 0.0699z + 0.2122}$$

A plot of the magnitude of the error is shown in Figure 6-4. A simple choice of the weight is given by $W(z) = 0.7$.

6.6 Conclusion

In this chapter a new H_∞-identification scheme is presented. The nominal models are obtained as the optimal H_∞-norm approximation of the system described by the measured frequency response samples. The problem is solved by a modified version of Algorithm 3.7. There are two main advantages of the proposed algorithm over existing H_∞-identification schemes. The obtained nominal models are rational transfer functions of the desired order and they are usually very good representatives of the true models.

Our experience with the algorithm suggests that for r sufficiently large, ε_{min} will be close to the measurement noise level ϵ. For sufficiently large N and r the worst case error bound can be close to 2ϵ which is the lowest possible bound for all H_∞-identification algorithms. To obtain a suitable order for the nominal model, the following strategy is suggested. Start with a low order r, and obtain the nominal model and the bound. Gradually increase the order until the error bound is within an acceptable level or until no significant reduction is achieved.

A general H_∞-identification algorithm is proposed. The algorithm provides a nominal model and frequency-dependent weighting function for the approximation error. Several examples are used to demonstrate the algorithms.

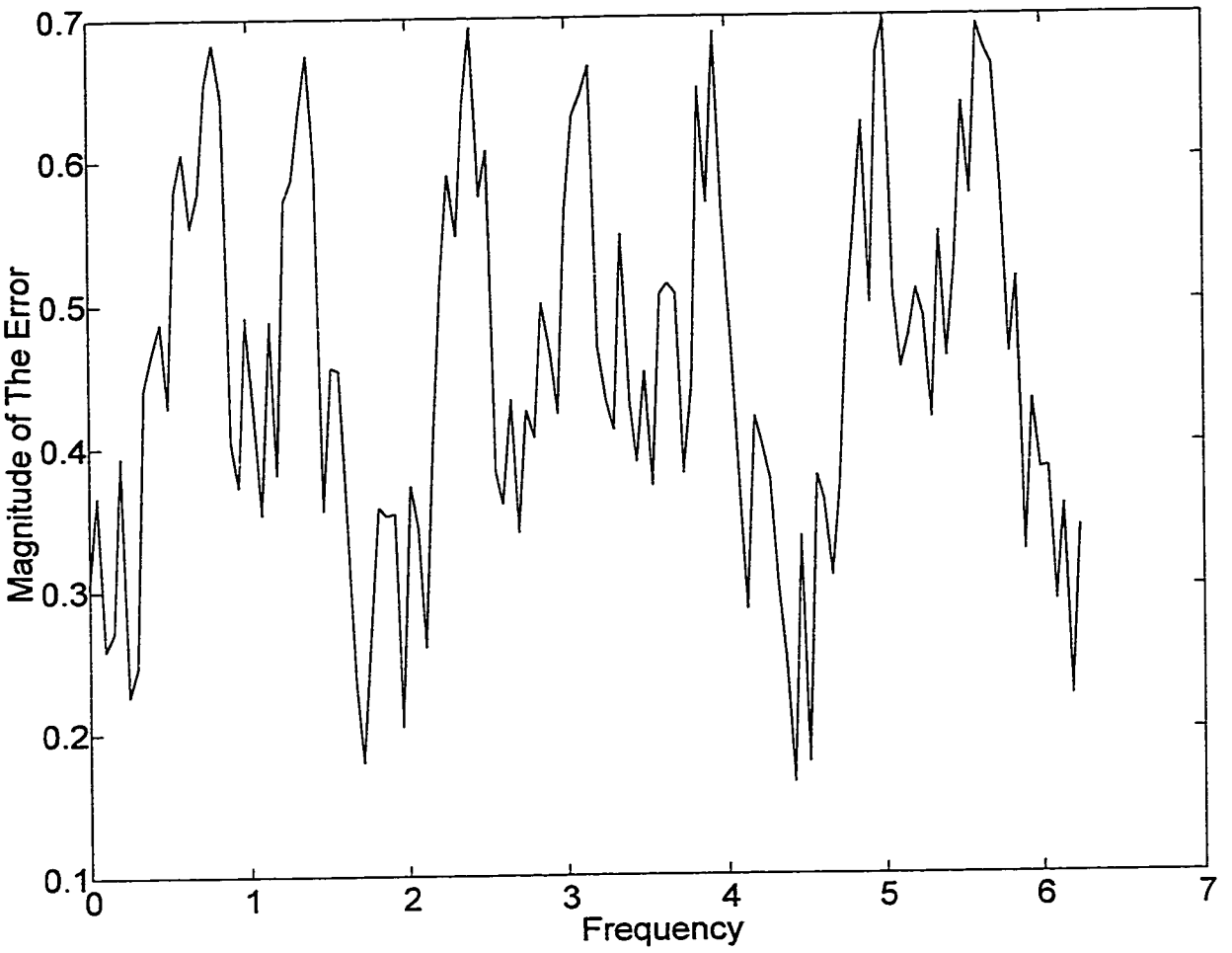

Figure 6-3: Plot of the Error Magnitude-Example 6.4

Chapter 7

H_∞-SIMULTANEOUS APPROXIMATION

7.1 Introduction

The simultaneous approximation problem is to find a single function that approximates a family of functions. The H_∞-norm approximation problem discussed earlier can be viewed as a special case of the simultaneous approximation problem where the family of functions to be approximated contains a single function only. In this chapter, a new approach to solve the H_∞-simultaneous approximation problem is proposed. The general approach is to convert the problem into a weighted H_∞-norm approximation problem. In Section 7.2, the simultaneous approximation problem is defined. The proposed solutions are given in Section 7.3. In Section 7.4, several problems from control theory are formulated as H_∞-simultaneous approximation problems and several case studies are given in Section 7.5. Throughout the chapter only SISO systems will be considered.

7.2 Simultaneous Approximation Problem

The simultaneous approximation problem (SAP) is concerned with finding a single function f^* that best approximates a family of functions F. This problem is well-known in mathematical literature. The definition of the problem is given next.

Definition 14 *Given a family of functions $F = \{f_1, f_2, ..., f_M\}$ defined on a compact metric space X, find a function f^* in a set of functions P such that*

$$\inf_{p \in P} \sup_{f \in F} \|p - f\| = \sup_{f \in F} \|f^* - f\|. \tag{7.1}$$

The above problem was investigated for the cases of real and complex valued functions. Necessary and sufficient conditions for the case when P is a set of polynomials (or linear combination of known functions) have been developed [117-120]. In [120], it was shown that a function f^* is the best simultaneous approximation to M functions in the L_2 norm if it is the best approximation to the mean value $\frac{1}{M}\sum_{i=1}^{M} f_i$. If the L_p norm is used, then the best simultaneous approximation is shown to be related to a weighted average in a more complicated manner. In many control applications, the ∞-norm is considered to be the most relevant measure of the error. When the ∞-norm is used, the best simultaneous approximation f^* is known as the Chebyshev center and the minimum value of (7.1) is known as the Chebyshev radius of the set. The Chebyshev simultaneous approximation is more complicated than the L_2 simultaneous approximation. For the case $F = \{f_1, f_2\}$, the problem of the Chebyshev simultaneous approximation of f_1 and f_2 was shown in [116] to be equivalent to approximating $\frac{f_1+f_2}{2}$ with the additive weight function $\frac{1}{2}|f_1 - f_2|$. In this chapter, the H_∞ SAP will be

considered. The problem is defined below.

Definition 15 *Given $G^1(z), G^2(z), ..., G^M(z)$ all in RH_∞. Let the corresponding weighting functions be $W^1(z), W^2(z), ..., W^M(z) \in RH_\infty$. Find an r^{th} order $G_r(z) \in RH_\infty$ such that*

$$\max_i \|W^i(G^i - G_r)\|_\infty \tag{7.2}$$

is minimized.

Simultaneous approximation was brought to the attention of the control community in [121]. In [121], the problem was formulated in state space and an LMI-based algorithm was proposed to solve it. To my knowledge this is the only algorithm that provides a solution to the L_∞ SAP. The algorithm in [121], works well if A_r and C_r are given but the solution to the general case is not as good. In some cases the algorithm may not converge, and if it converges it may converge to a local minimum. In this chapter we present a new approach to solve the SAP. The algorithm is based on H_∞-norm approximation algorithms developed in Chapter 3.

7.3 Solution of the Simultaneous Approximation Problem

In this section a scheme to solve the general SAP is presented. The general approach used in the thesis is to convert the problem into a weighted H_∞-norm approximation problem which can be solved by the frequency domain algorithm presented in Chapter 3. The fact that the above algorithm uses a finite set of frequency response samples makes the proposed simultaneous approximation algorithms possible.

In all the algorithms presented in this section, we assume the availability of a set of N

frequencies $\Omega = \{\omega_1, \omega_2, ..., \omega_N\}$ and samples of $G^i(z)$ and the corresponding weights $W^i(z)$ at $z = e^{j\omega_k}$ where $\omega_k \in \Omega$.

The motivation for this approach comes from graphical interpretation of the problem and the main idea is illustrated by the following unweighted case. The frequency response of the systems G^i are complex functions defined on $\omega_k \in \Omega$. The simultaneous approximation problem can be interpreted as: The best simultaneous approximation is selected as the one that has a frequency response that is at the smallest possible distance from the frequency responses of G^i at each frequency. To obtain a suboptimal solution one can first obtain an empirical transfer function defined on Ω, and then approximate it by a transfer function of the desired order.

Before attempting to solve the general problem, two important special cases will be considered. The single system case will be considered in Section 7.3.1 and the unweighted S.A.P will be treated in Section 7.3.2. An algorithm to solve the general problem will be presented in 7.3.3.

7.3.1 The Single System Case

In this section we consider the special case where $G^1 = G^2 = \cdots = G^M = G$. The problem is to find G_r of the desired order that minimizes

$$\max_{i \in [1,M]} \|W^i(G - G_r)\|_\infty. \tag{7.3}$$

This special case may arise in several applications, an example of which is the controller reduction problem where the closed-loop system is required to satisfy several performance specifications. Discretizing (7.3) leads to the following

$$\min_{G_r \in RH_\infty} \max_{i\in[1,M]} \max_{\omega_k \in \Omega} |W^i(e^{j\omega_k})(G(e^{j\omega_k}) - G_r(e^{j\omega_k}))| . \tag{7.4}$$

Defining $W(e^{j\omega_k})$ as

$$W(e^{j\omega_k}) = \max_{i\in[1,M]} |W^i(e^{j\omega_k})| \quad \text{for } \omega_k \in \Omega. \tag{7.5}$$

The problem (7.4) is equivalent to

$$\min_{G_r} \max_{w_k} |W(e^{j\omega_k}) \, (G(e^{j\omega_k}) - G_r(e^{j\omega_k}))| \tag{7.6}$$

which is a discretized version of the H_∞ -norm model reduction problem and it can be solved by Algorithm 3.8. It may be difficult to find an analytical expression for $W(z)$ satisfying (7.5), but this is not needed by Algorithm 3.8. In summary, the solution to the single system problem is given by the following algorithm.

***ALGORITHM* 7.1: The Single System SAP**

Given $G(e^{j\omega_k}), W^i(e^{j\omega_k})$ for $i \in [1, M]$ and $\omega_k \in \Omega$ and the desired order r,

Step 1: Obtain $W(e^{j\omega_k})$ for $\omega_k \in \Omega$, using

$$W(e^{j\omega_k}) = \max_{i\in[1,M]} |W^i(e^{j\omega_k})|$$

Step 2: Use Algorithm 3.8 to obtain an r^{th} order transfer function G_r that minimizes (7.6).

7.3.2 The Unweighted Case

In this section a slightly more difficult special case will be considered. This is the unweighted case ($W^i = 1$, for $i \in [1, M]$). The problem to be solved can be stated as follows: Find G_r of

the desired order such that

$$\min_{G_r \in RH_\infty} \max_{i \in [1,M]} \|G^i - G_r\|_\infty. \tag{7.7}$$

Discretizing the problem (7.7) leads to the following problem

$$\min_{G_r \in RH_\infty} \max_{i \in [1,M]} \max_{\omega_k \in \Omega} \left| G^i(e^{j\omega_k}) - G_r(e^{j\omega_k}) \right|. \tag{7.8}$$

Note that for any $\hat{G}(e^{j\omega_k})$, the following hold

$$\begin{aligned} \min_{G_r \in RH_\infty} \max_{i \in [1,M]} \max_{\omega_k \in \Omega} |G^i(e^{j\omega_k}) - G_r(e^{j\omega_k})| \;\leq\; & \min_{\hat{G}} \max_{i \in [1,M]} \max_{\omega_k \in \Omega} |G^i(e^{j\omega_k}) - \hat{G}(e^{j\omega_k})| \\ & + \min_{G_r \in RH_\infty} \max_{\omega_k \in \Omega} |\hat{G}(e^{j\omega_k}) - G_r(e^{j\omega_k})| \end{aligned} \tag{7.9}$$

The approach that will be taken here is divided into two steps. First obtain an empirical transfer function $\hat{G}$ defined on the discrete frequency set Ω by solving

$$\min_{\hat{G}} \max_{i \in [1,M]} \max_{\omega_k \in \Omega} |G^i(e^{j\omega_k}) - \hat{G}(e^{j\omega_k})|$$

for $\hat{G}(e^{j\omega_k})$ at each frequency. This can be viewed as follows. At each frequency, find the smallest circle containing the M complex numbers $G^i(e^{j\omega_k})$. Note that $\hat{G}(e^{j\omega_k})$ obtained in this step need not be expressed as an analytical transfer function. Instead an empirical transfer function defined at $\omega_k \in \Omega$ is obtained. The second step is to approximate $\hat{G}(e^{j\omega_k})$ by $G_r(e^{j\omega_k})$ of the desired order. The algorithm is outlined below.

***ALGORITHM* 7.2: A Sub-Optimal Solution to the Unweighted SAP**

Given $G^i(e^{j\omega_k})$, for $i \in [1, M]$ and $\omega_k \in \Omega$ and the desired order r,

Step 1: Obtain an empirical transfer function $\hat{G}(e^{j\omega_k})$ by solving the following N problems ($k \in [1, N]$)

$$\min \gamma_k$$

subject to

$$\left|\hat{G}(e^{j\omega_k}) - G^i(e^{j\omega_k})\right| \leq \gamma_k$$

Step 2: Using Algorithm 3.7, obtain an r^{th} order transfer function G_r that minimizes

$$\|\hat{G} - G_r\|_\infty$$

Remark 5 *A frequency dependent lower bound on the achievable error is given by* γ_k.

Remark 6 *The algorithm can be easily extended to the case of identical weights* ($W^i(z) = W(z)$ *for* $i \in [1, M]$). *In such a case Step 2 is replaced by*

Step 2':Use Algorithm 3.8 to find an r^{th} *order* G_r *that minimizes*

$$\|W(\hat{G} - G_r)\|_\infty.$$

Remark 7 *In Step 1, each problem is equivalent to finding a minimum radius disk containing* M *points. In the special case* $M = 2$, $\hat{G}$ *can be obtained directly as*

$$\hat{G}(e^{j\omega_k}) = \frac{1}{2}\left(G^1(e^{j\omega_k}) + G^2(e^{j\omega_k})\right)$$

and

$$\gamma_k = \frac{1}{2} \left| G^1(e^{j\omega_k}) - G^2(e^{j\omega_k}) \right|.$$

Approximating $\hat{G}(e^{j\omega_k})$ leads to the simultaneous approximation if it is possible to get G_r such that $\|\hat{G} - G_r\|_\infty = 0$, which may not be possible in general. The above algorithm can give a sub-optimal solution to the unweighted simultaneous approximation problem. The following algorithm attempts to solve the optimal unweighted simultaneous approximation problem.

***ALGORITHM* 7.3: The Unweighted SAP (An Improved Algorithm)**

Given $G^i(e^{j\omega_k})$ for $i \in [1, M]$ and $\omega_k \in \Omega$, $l_{\max}$ and the desired order r.

Step 1: Obtain an empirical transfer function $\hat{G}(e^{j\omega_k})$ by solving the following N problems

$$\min \gamma_k$$

subject to

$$\left| \hat{G}(e^{j\omega_k}) - G^i(e^{j\omega_k}) \right| \leq \gamma_k$$

for $i \in [1, M]$, $k \in [1, N]$ and $\omega_k \in \Omega$.

Step 2: Set $l = 1$, $U^l(e^{j\omega_k}) = 1$ for $\omega_k \in \Omega$

Step 3: Solve for $G_r = \frac{n_r(e^{j\omega_k})}{d_r(e^{j\omega_k})}$ from

$$\min_{n_r(e^{j\omega_k}),\, d_r(e^{j\omega_k})} \sum_{l=1}^{N} \left| \left(d_r(e^{j\omega_k}) \hat{G}(e^{j\omega_k}) - n_r(e^{j\omega_k}) \right) U^l(e^{j\omega_k}) \right|^2$$

Step 4: Compute the weighted error $E(e^{j\omega_l})$

$$E(e^{j\omega_k}) = \max_{i\in[1,M]} \max_{\omega_k\in\Omega} |G^i(e^{j\omega_k}) - G_r(e^{j\omega_k})|$$

Step 5: Update the weight according to

$$U^{l+1}(e^{j\omega_k}) = E(e^{j\omega_k})U^l(e^{j\omega_k})/\alpha \text{ where } \alpha \text{ is a scaling factor.}$$

Step 6: If $l = l_{\max}$, stop. Otherwise set $l = l + 1$ and go to Step 3.

Remark 8 *In this algorithm the actual simultaneous approximation error is used in updating the weight. This is expected to produce better reduced order simultaneous approximations. Note also that if $M = 1$, then this is exactly Algorithm 3.7.*

7.3.3 The General S.A.P

In this section, an algorithm to solve the general SAP will be proposed. A sub-optimal algorithm consists of two parts. In the first part an empirical transfer function is obtained and then a weighted H_∞-norm approximation problem is solved to obtain the best simultaneous approximation of the desired order. The algorithm is summarized as follows.

***ALGORITHM* 7.4: Sub-Optimal Solution to the General SAP**

Given $G^i(e^{j\omega_k}), W^i(e^{j\omega_k})$ for $i \in [1, M]$ and $\omega_k \in \Omega$ and the desired order r .

Step 1: Obtain an empirical transfer function $\hat{G}(e^{j\omega_k})$ by solving the following N problems

$$\min \gamma_k$$

subject to

$$\left|W^i(e^{j\omega_k})(\hat{G}(e^{j\omega_k}) - G^i(e^{j\omega_k}))\right| \leq \gamma_k$$

for $i \in [1, M]$, $k \in [1, N]$ and $\omega_k \in \Omega$.

Step 2: Solve the single system case using Algorithm 7.1 to find an r^{th} order transfer function $G_r = \frac{n_r(e^{j\omega_k})}{d_r(e^{j\omega_k})}$ that minimizes

$$\|W^i(\hat{G} - G_r)\|_\infty$$

Obtaining an r^{th} order approximation to the above empirical transfer function does not necessarily produce the best simultaneous approximation. In the remaining part of this section an improved simultaneous approximation algorithm is developed.

***ALGORITHM* 7.5: The General SAP**

Given $G^i(e^{j\omega_k}), W^i(e^{j\omega_k})$ for $i \in [1, M]$ and $\omega_k \in \Omega$, $l_{\max}$ and the desired order r.

Step 1: Obtain an empirical transfer function $\hat{G}(e^{j\omega_k})$ by solving the following N problems

$$\min \gamma_k$$

subject to

$$\left|W^i(e^{j\omega_k})\left(\hat{G}(e^{j\omega_k}) - G^i(e^{j\omega_k})\right)\right| \leq \gamma_k$$

for $i \in [1, M]$, $k \in [1, N]$ and $\omega_k \in \Omega$.

Step 2: Set $l = 1$, $U^l(e^{j\omega_k}) = 1$ for $\omega_k \in \Omega$

Step 3: Solve for $G_r = \frac{n_r(e^{j\omega_k})}{d_r(e^{j\omega_k})}$ from

$$\min_{n_r(.),\, d_r(.)} \sum_{l=1}^{N} \left| \left(d_r(e^{j\omega_k}) G(e^{j\omega_k}) - n_r(e^{j\omega_k}) \right) W(e^{j\omega_k}) U^l(e^{j\omega_k}) \right|^2$$

Step 4: Compute the weighted error $E(e^{j\omega_l})$

$$E(e^{j\omega_k}) = \max_{i\in[1,M]} \max_{\omega_k \in \Omega} \left| G^i(e^{j\omega_k}) - G_r(e^{j\omega_k}) \right|$$

Step 5: Update the weight using

$$U^{l+1}(e^{j\omega_k}) = E(e^{j\omega_k}) W(e^{j\omega_k}) U^l(e^{j\omega_k})/\alpha \quad \text{where } \alpha \text{ is a scaling factor.}$$

Step 6: If $l = l_{\max}$, stop. Otherwise set $l = l + 1$ and go to Step 3.

Remark 9 *In this algorithm the actual simultaneous approximation error is used in updating the weight. This is expected to produce better reduced order simultaneous approximations.*

7.4 Applications of the H_∞ Simultaneous Approximation Problems

Many control systems problems can be formulated as simultaneous approximation problems. A brief list of some possible applications is given below [121].

1. Controller Reduction: A high order controller may be designed to achieve stability and performance. A weighted approximation problem can be formulated to obtain a reduced order controller satisfying a given specification (see Chapter 5). One can find $W^1(e^{j\omega_k})$

such that if

$$\left\|W^1(K_n - K_r)\right\|_\infty < 1$$

then that particular specification is satisfied. If there are M specifications to be satisfied, then one can find $W^i(e^{j\omega_k})$ for $i \in [1, M]$. To satisfy them all one needs

$$\max_{i\in[1,M]} \left\|W^1(K_n - K_r)\right\|_\infty < 1$$

This is an example of the single system case and can be solved by Algorithm 7.1. An alternative approach is to design M controllers to satisfy M different specifications and then try to simultaneously approximate them by a single controller.

2. Linearization of Nonlinear Systems: For a given nonlinear system and an operating point, one can obtain an approximation to the original system by linearizing about the given operating point. Now consider the problem of finding a linear model that approximates the system at M operating points. Such a problem can be formulated as an S.A.P. The nominal model can be obtained as follows: Select a set of M operating points and linearize the nonlinear system at each operating point to obtain $G^i(e^{j\omega_k})$. Now obtain $G_r(e^{j\omega_k})$ to simultaneously approximate the M models. This is an example of the unweighted SAP.

3. Nominal Model Selection: Consider a system with parametric uncertainty. For a given set of M combinations of the parameters one can obtain M models describing the process. A nominal model may be obtained to minimize the uncertainty bound by simultaneously approximating the M models. For systems with failing parts, a nominal model can be obtained as follows. The set of possible systems are obtained by considering all fail-

ure modes, and then obtaining a single system that approximates the family of possible systems.

4. Stochastic Modeling: The simultaneous approximation can be considered as the "best" representative of an infinite number of realizations of the stochastic system.

7.5 Examples

In this section we apply the algorithms of Section 7.3 to several examples from the available literature.

7.5.1 Example 7.1: Four Disk Problem

In this example we consider the well-known four disk problem [73]. The transfer function is given by

$$G(s) = \frac{0.0064432s^5 + 0.0023196s^4 + 0.071252s^3 + 1.0002s^2 + 0.10455s + 0.99551}{s^8 + 0.161s^7 + 6.004s^6 + 0.58215s^5 + 9.9835s^4 + 0.40727s^3 + 0.3982s^2}$$

In [72], it was shown that if K is a stabilizing controller then the reduced order controller K_r, having the same number of unstable poles as K stabilizes the system provided that

$$\|W^1(K - K_r)\|_\infty < 1$$

where

$$W^1 = (I + GK)^{-1}G.$$

It is also shown that K_r preserves the performance if

$$\|W^2(K - K_r)\|_\infty < 1$$

where

$$W^2 = W^1(I + GK)^{-1}.$$

A meaningful controller reduction scheme is to select the reduced order controller K_r that simultaneously satisfies both conditions. In other words, minimize $\gamma_0 = max\{\|W^1(K - K_r)\|_\infty, \|W^2(K - K_r)\|_\infty\}$. Then $\gamma_0 < 1$ implies that K_r will guarantee both stability and performance. Algorithm 7.1 was used to obtain reduced order controllers of different orders and the result is summarized in Table 7.1.

TABLE 7.1: Reduced Order Controller for Example 7.1

Order	$\max_i \|W^i(K^i - K_r)\|_\infty$	K_r
0	1.1737	0.0355
1	0.7420	$\frac{0.0932s+0.0052}{s+0.2176}$
2	0.4123	$\frac{0.0349s^2+0.0213s+0.0012}{s^2+0.2263s+0.0635}$
3	0.3115	$\frac{0.0462s^3+0.0079s^2+0.0335s+0.0014}{s^3+0.4258s^2+0.3618s+0.0769}$
4	0.2695	$\frac{0.0316s^4+0.0552s^3+0.0246s^2+0.0369s+0.0018}{s^4+0.5135s^3+0.4821s^2+0.3149s+0.0836}$

From Table 7.1, it is clear that reduced order controller K_r simultaneously satisfies stability and performance requirements for all $r \geq 1$.

7.5.2 Example 7.2: Lateral Autopilot Model

Consider the lateral autopilot model studied in [121]. The transfer functions are given by

$$G_1(s) = \frac{129.54s^3 + 604.4s^2 + 2167s + 2197}{s^5 + 31.046s^4 + 272.122s^3 + 1419.9s^2 + 7930.5s + 276.7}$$

$$G_2(s) = \frac{161.9s^3 + 906.9s^2 + 2689s + 2561}{s^5 + 31.046s^4 + 272.122s^3 + 1419.9s^2 + 7930.5s + 276.7}$$

$$G_3(s) = \frac{33.1s^3 - 297.09s^2 + 610.2s + 111.4}{s^5 + 31.046s^4 + 272.122s^3 + 1419.9s^2 + 7930.5s + 276.7}$$

We consider approximating the three transfer functions by a single fifth order transfer function. Since the denominators of all three transfer functions are the same, a reasonable assumption is to leave the denominator unchanged and obtain the best numerator. Algorithm 7.2 was used to obtain the optimal numerator.

$$-5.3376 \times 10^{-7}s^5 + 4.7753 \times 10^{-6}s^4 + 97.4998s^3 + 304.9045s^2 + 1649.6s + 1836.2$$

with an approximation error $\gamma = 2.6194$ which is the same value reported in [121]. Algorithm 7.3 was used to obtain simultaneously approximating transfer functions of lower orders. The result is summarized in Table 7.2.

TABLE 7.2: Optimal Simultaneous Approximation for Example 7.2

Order	$\max_i \|(G^i - G_r)\|_\infty$	G_r
1	2.6228	$\frac{1.4106s+0.2591}{s+0.0391}$
2	2.6259	$\frac{1.0299s^2+21.7484s+3.3318}{s^2+17.2153s+0.5016}$
3	2.6194	$\frac{0.1686s^3+1.8704s^2+6.9570s+9.4139}{s^3+1.0502s^2+40.4318s+1.4186}$

From Table 7.2, one can observe that there is no significant reduction in error level by using models of higher orders. A third order simultaneously approximating model is almost as good as the fifth order one.

7.5.3 Example 7.3: Nominal Model Selection

In this example, the problem of determining a nominal model is solved by first posing the problem as an SAP and then using Algorithm 7.2 to solve it. Consider the following system

$$G = \frac{z^2 - 1.1z + 0.24}{z^2 - 1.6z + 0.68}$$

Assume that two sets of frequency response measurements are available. The nominal model of the required order r is obtained as the r^{th} order best simultaneous approximation to the two sets of measurements. The true frequency response is generated using 1024 points on the unit circle. The simulated frequency response measurements are produced from the true values by corrupting them with additive noise with magnitude less than or equal to 0.1. The best approximation of orders $1, 2$ and 3 are given in Table 7.3. The third column of the table provides the Chebyshev radius of the set containing the two simulated transfer functions. The actual ∞-norm of the error is given in the fourth column.

TABLE 7.3: Nominal Models and the Corresponding Error for Example 7.3

Order	G_r	$max_i \|(G^i - G_r)\|_\infty$	$\|(G - G_r)\|_\infty$
1	$\frac{1.4082z-0.7946}{z-0.7442}$	0.7093	0.6486
2	$\frac{1.0002z^2-1.0952z+0.2402}{z^2-1.5931z+0.6754}$	0.0971	0.0275
3	$\frac{1.0008z^3-0.3924z^2-0.5346z+0.1752}{z^3-0.8891z^2-0.4495+0.4797}$	0.0969	0.0221

It can be observed that the approximation error for the second and the third order approximations are approximately equal to the measurement noise level, which indicates that this is the optimal solution or very close to it. Note also that increasing the order from two to three has no significant effect on the error which indicates that a second order is probably a good choice for the order.

7.6 Conclusions

In this chapter we considered the H_∞ simultaneous approximation problem. In the single system case, the problem can be easily converted to a weighted H_∞ approximation problem. Another algorithm to solve the unweighted simultaneous approximation problem was presented in Section 7.3.2. Algorithms to solve the general problem are proposed in Section 7.3.3. Case studies in controller reduction and nominal model selection were solved using the developed algorithm.

Chapter 8

APPROXIMATION IN 2-D

8.1 Introduction

Approximation of two-dimensional systems has many applications such as 2-D filter design and image compression. The approximation problem in 2-D is much more complicated than its 1-D counterpart. An obvious source of difficulty is the considerable increase in the data to be handled. A typical 2-D application may involve processing millions of data points per second. Simple extensions of the 1-D method to handle the 2-D problems become prohibitively complex and impractical. There are other difficulties that come from the nature of the 2-D problem. Some 2-D systems are best represented by partial difference equations, which are more difficult to handle compared to the ordinary difference equations in the case of 1-D. Another source of difficulty is the lack of a fundamental theorem of algebra in the 2-D case. This creates difficulties in factoring and analyzing 2-D systems.

In this chapter, Algorithm 3.7 is extended to solve the 2-D approximation problem. In Section 8.2, preliminary material on 2-D systems is provided. The 2-D approximation problem

and the existing techniques to solve it are presented in Section 8.3. Least squares identification of 2-D systems is summarized in Section 8.4. In Section 8.5, we proposed a new algorithm and in Section 8.6 illustrative examples are given.

8.2 Preliminary Material on 2-D Systems

In this section, preliminary material on two-dimensional systems are given. Linear shift invariant 2-D systems can be described by 2-D complex functions. The 2-D rational transfer functions are usually expressed in the following form:

$$G_n(z_1, z_2) = \frac{\sum\limits_{(i,j)\in R_b^o} b_{ij} z_1^{-i} z_2^{-j}}{\sum\limits_{(i,j)\in R_a^o} a_{ij} z_1^{-i} z_2^{-j}}$$

where R_b^o, R_a^o are the region of support for the numerator and denominator polynomials. The coefficients $b_{ij} = 0\ \forall (i,j) \notin R_b^o$ and similarly $a_{ij} = 0\ \forall (i,j) \notin R_a^o$.

Definition 16 *The 2-norm of a 2-D system $G(z_1, z_2)$ is defined as*

$$\|G(z_1, z_2)\|_2 = \left(\int_0^{2\pi} \int_0^{2\pi} \left| G(e^{j\omega_1}, e^{j\omega_2}) \right|^2 d\omega_1 d\omega_2 \right)^{1/2} \tag{8.1}$$

Definition 17 *The ∞-norm of a 2-D system $G(z_1, z_2)$ is defined as*

$$\|G(z_1, z_2)\|_\infty = \sup_{\omega_1 \in \mathcal{R}} \sup_{\omega_2 \in \mathcal{R}} \left| G(e^{j\omega_1}, e^{j\omega_2}) \right| \tag{8.2}$$

Definition 18 *A shift invariant system $G(z_1, z_2)$ is separable if it can be expressed as*

$$G(z_1, z_2) = G_1(z_1)G_2(z_2).$$

Separable systems are very special cases of 2-D systems and most of the results on 1-D systems extend easily to them. Another special case involves the separable denominator systems where the denominator is separable but the numerator is not.

8.3 Existing Algorithms to Solve the 2-D Approximation Problem

In this section the approximation of 2-D systems is considered. The 2-D approximation problem can be stated as: Given an original 2-D transfer function $G_n(z_1, z_2)$ of the form:

$$G_n(z_1, z_2) = \frac{\sum\limits_{(i,j)\in R_b^o} b_{ij} z_1^{-i} z_2^{-j}}{\sum\limits_{(i,j)\in R_a^o} a_{ij} z_1^{-i} z_2^{-j}}$$

where R_b^o, R_a^o are the region of support for the numerator and denominator polynomials. The problem is to approximate the above system by a system of the following form

$$G_r(z_1, z_2) = \frac{n_r(z_1, z_2)}{d_r(z_1, z_2)} = \frac{\sum\limits_{(i,j)\in R_b} b_{ij}\ z_1^{-i} z_2^{-j}}{1 + \sum\limits_{(i,j)\in R_a} a_{ij}\ z_1^{-i} z_2^{-j}}$$

where R_b, R_a are the region of support corresponding to n_r and d_r respectively. Throughout

this work, we will be mainly interested in transfer functions having the form

$$G_r(z_1, z_2) = \frac{\sum_{i=0}^{M_1} \sum_{j=0}^{M_2} b_{ij} z_1^{-i} z_2^{-j}}{\sum_{i=0}^{N_1} \sum_{j=0}^{N_2} a_{ij} z_1^{-i} z_2^{-j}} \quad .$$

2-D model reduction has been studied by many researchers and several techniques have been proposed. 2-D Pade's approximation was one of the earliest techniques to be used [157, 148]. As in the 1-D case, the basic idea here is to match impulse response coefficients of both the original and the approximate models. Using the matching conditions it is possible to write a set of linear equations in the unknown parameters of the reduced order model. There are two major problems with Pade's technique. First, the size of the region in which the coefficients are matched depends on the order of the approximate model. If one wants to increase this region, then the order of the approximate model needs to be increased. The second problem is that stability of the approximate model is not preserved.

Prony's methods have also been used for approximating 2-D systems [157] . The impulse response is matched over larger regions but as with Pade's approximation, the stability is not necessarily preserved.

State-space based model reduction have been extended to the 2-D case. The state space representation has been generalized to the 2-D case and there are several 2-D state space models used in literature. These include: Roesser's model, Attasi's models and Fornasini-Marchesini models. Unlike the 1-D case, there are several accepted definitions of balanced realizations that are not necessarily equivalent in the 2-D case and as a result several balanced truncation algorithms are possible. Lu, Lee and Zhang [159], introduced the 'pseudo-balanced' realization.

Wang, Glover and Doyle [141] proposed the "structurally-balanced" realization . Zhou et al proposed the "quasi-balanced" realization [122]. Error bounds are available, but it was shown that direct truncation from pseudo-balanced or quasi-balanced realizations does not guarantee the stability of the reduced order model [122].

8.4 Identification of 2-D Systems

The least squares method is widely used to identify two-dimensional systems because of its simplicity. In this section, a summary of 2-D least squares identification is presented. The 2-D systems considered in this work are rational in general. In this work the approximate model is assumed to have the following form:

$$G_r(z_1, z_2) = \frac{\sum_{i=0}^{M_1} \sum_{j=0}^{M_2} b_{ij} z_1^{-i} z_2^{-j}}{\sum_{i=0}^{N_1} \sum_{j=0}^{N_2} a_{ij} z_1^{-i} z_2^{-j}} \quad .$$

with $a_{00} = 1$. The LS algorithm can be used to determine the unknown coefficients to minimize

$$\left\| G_n(z_1, z_2) - \frac{\sum_{i=0}^{M_1} \sum_{j=0}^{M_2} b_{ij} z_1^{-i} z_2^{-j}}{\sum_{i=0}^{N_1} \sum_{j=0}^{N_2} a_{ij} z_1^{-i} z_2^{-j}} \right\|_2$$

As discussed earlier, $G_r(z_1, z_2)$ cannot be represented as a linear combination of any bases. Therefore the modified LS technique will be used. The unknown parameters are determined to minimize

$$\left\| \left(\sum_{i=0}^{N_1} \sum_{j=0}^{N_2} a_{ij}\, z_1^{-i} z_2^{-j} \right) G_n(z_1, z_2) - \left(\sum_{i=0}^{M_1} \sum_{j=0}^{M_2} b_{ij}\, z_1^{-i} z_2^{-j} \right) \right\|_2$$

which can be represented in the form

$$\| G_n(z_1, z_2) - \Phi\theta \|_2 .$$

This is a linear approximation algorithm and the existence and uniqueness of the solution to the above problem was proved in [158].

8.5 The Proposed Algorithm

In this section, Algorithm 3.7 is extended to the 2-D case. It is assumed that the original 2-D transfer function $G_n(z_1, z_2)$ is of the form

$$G_n(z_1, z_2) = \frac{\sum\limits_{(i,j)\in R_b^o} b_{ij} z_1^{-i} z_2^{-j}}{\sum\limits_{(i,j)\in R_a^o} a_{ij} z_1^{-i} z_2^{-j}},$$

where R_b^o, R_a^o are the regions of support for the numerator and denominator polynomials. It is required to approximate the above system by a system having the following form:

$$G_r(z_1, z_2) = \frac{n_r(z_1, z_2)}{d_r(z_1, z_2)} = \frac{\sum\limits_{(i,j)\in R_b} b_{ij}\, z_1^{-i} z_2^{-j}}{1 + \sum\limits_{(i,j)\in R_a} a_{ij}\, z_1^{-i} z_2^{-j}}.$$

The frequency domain algorithm in Section 3.8 is extended to solve the above approximation

problem. The main idea is the same. However implementation of it becomes more involved.

***ALGORITHM* 8.1: 2-D H_∞-Norm Model Reduction**

Given $G(z_1, z_2) \in RH_\infty$, R_a, R_b, N_1, N_2 and l_{max}.

Step 1: Discretize $G(z_1, z_2)$ on $N_1 \times N_2$ points $z_1 = e^{j\omega_{1k}} = e^{j\frac{2\pi k_1}{N_1}}$, $z_2 = e^{j\omega_{2k}} = e^{j\frac{2\pi k_2}{N_2}}$. Let $l = 1$ and set $U^l(e^{j\omega_{1k}}, e^{j\omega_{2k}}) = 1$

Step 2: Solve for $G_r(e^{j\omega_{1k}}, e^{j\omega_{2k}}) = \frac{n_r^l(e^{j\omega_{1k}}, e^{j\omega_{2k}})}{d_r^l(e^{j\omega_{1k}}, e^{j\omega_{2l}})}$ from

$$\min_{d_r,\, n_r} \|G_n(e^{j\omega_{1k}}, e^{j\omega_{2k}}) d_r^l(e^{j\omega_{1k}}, e^{j\omega_{2l}}) - n_r^l(e^{j\omega_{1k}}, e^{j\omega_{2k}}))U^l(e^{j\omega_{1k}}, e^{j\omega_{2k}})\|_2$$

Step 3: Update $U(e^{j\omega_{1k}}, e^{j\omega_{2k}})$

$$U^{l+1}(e^{j\omega_{1k}}, e^{j\omega_{2k}}) = \left|\left(G_n(e^{j\omega_{1k}}, e^{j\omega_{2k}}) - G_r^l(e^{j\omega_{1k}}, e^{j\omega_{2k}})\right)\, U^l(e^{j\omega_{1k}}, e^{j\omega_{2k}})\right| \,/\alpha$$

Step 4: If $l = l_{max}$, stop. Otherwise set $l = l + 1$ and go to Step 2.

The least squares problem in Step 2 can be expressed in the form (8.3) and standard least squares algorithms can be used to solve it. The FIR approximation can also be solved by the above algorithm. In such a case G_r is restricted to the following form:

$$G_r(e^{j\omega_{1k}}, e^{j\omega_{2k}}) = n_r^l(e^{j\omega_{1k}}, e^{j\omega_{2k}}).$$

8.6 Examples

8.6.1 Example 8.1: Zero-Phase FIR Filter

In this section, an example of a 2-D zero phase FIR filter is designed to satisfy the following specifications:

$$G_n(e^{jf_1}, e^{jf_2}) = \begin{cases} 1 & \sqrt{f_1^2 + f_2^2} \leq 0.2 \\ 0 & \sqrt{f_1^2 + f_2^2} \geq 0.6 \end{cases}$$

This example has been studied by Harris and Mersereau, Burnside and Parks [146] and others. In [146], the performance of different algorithms is compared for different filter sizes. Algorithm 8.1 was used to find 2-D filters of different orders. The models are assumed to be of the form:

$$G_r(z_1, z_2) = \sum_{i=-r}^{r} \sum_{j=-r}^{r} b_{ij} z_1^{-i} z_2^{-j}.$$

The peak error is given in Table 8.1 for Algorithm 8.1 together with other methods reported in the literature for different filter sizes.

Table 8.1: H_∞-norm Error Achieved by Different Designs of FIR Filters

Algorithm	5×5	7×7
Chi & Chiou	0.2718	0.1281
Algazi et al	0.2733	0.1336
Lee & Chen	-	0.1270
Harris	0.2767	0.1273
Burnside & Parks	0.2664	0.1263
Algorithm 8.1	0.2679	0.1284

This table shows that the algorithm leads to a filter comparable to the other methods. The

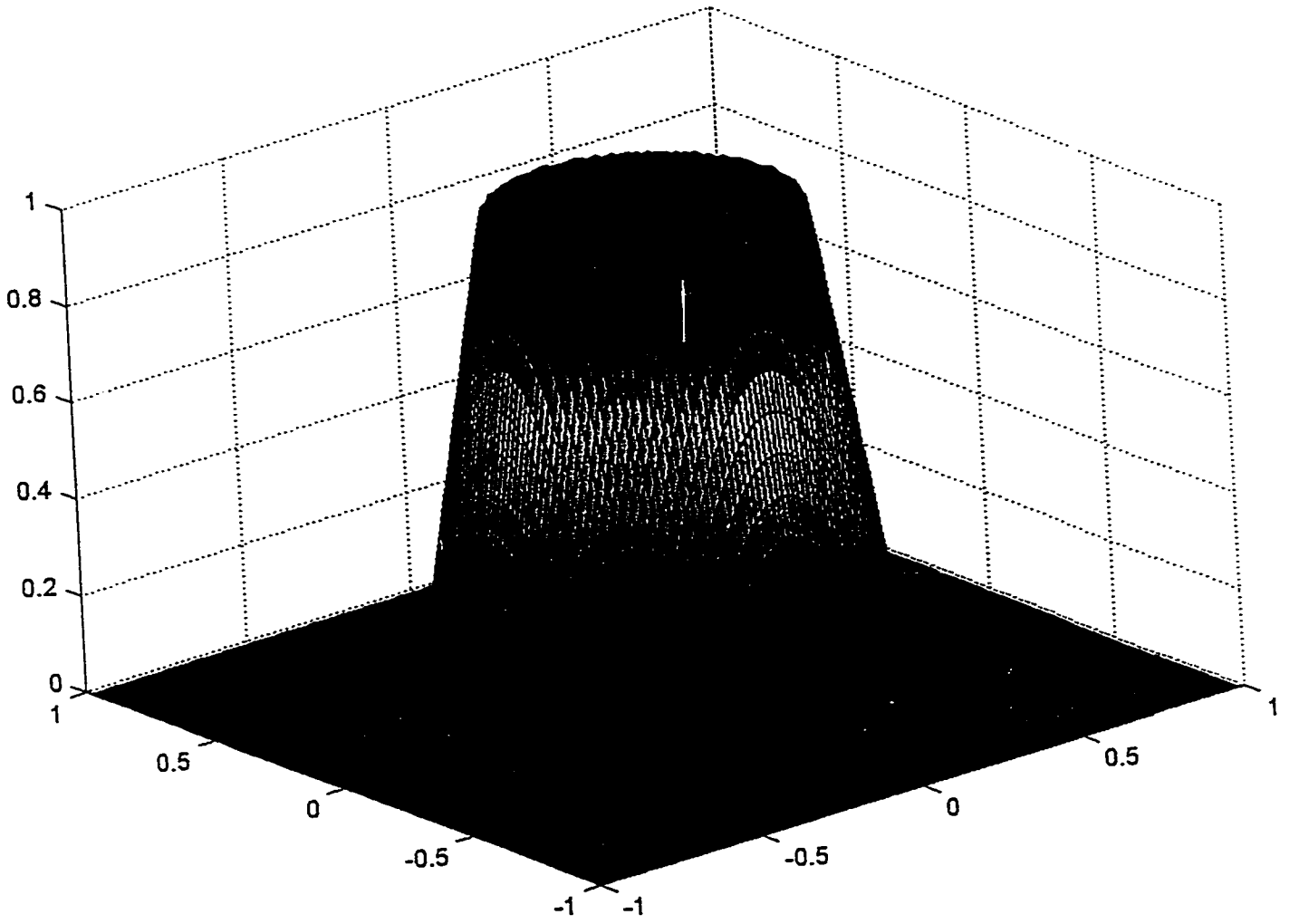

Figure 8-1: Desired Filter Characteristics

frequency response of the filter for 5×5 and 7×7 FIR filters are shown in Figures 8.2 and 8.3 respectively.

8.6.2 Example 8.2: 2D-IIR Filter

Consider the filter specifications given in Example 8.1. Algorithm 8.1 is used to obtain IIR filters. The filters are assumed to have the following form:

$$G_r(z_1, z_2) = \frac{\sum_{i=-r}^{r} \sum_{j=r}^{r} b_{ij} z_1^{-i} z_2^{-j}}{\sum_{i=-r}^{r} \sum_{j=-r}^{r} a_{ij} z_1^{-i} z_2^{-j}}$$

with $a_{00} = 1$. Filters with $r = 2$ and 3 are designed and their frequency responses are given in Figures 8.4 and 8.5 respectively. The obtained peak error is 0.2087 for $r = 2$ and 0.0498 for $r = 3$. The IIR filters provide better approximation of the desired filter compared to the FIR filters in Example 8.1. In Figure 8.5, only the response in passband and stopband are shown.

8.7 Summary

In this chapter, anew 2-D H_∞-norm approximation algorithm was proposed. The new algorithm is an extension of Algorithm 3.7. The algorithm was used to obtain FIR and IIR filters to approximate ideal 2-D filter specifications. The result is comparable to the result reported in literature.

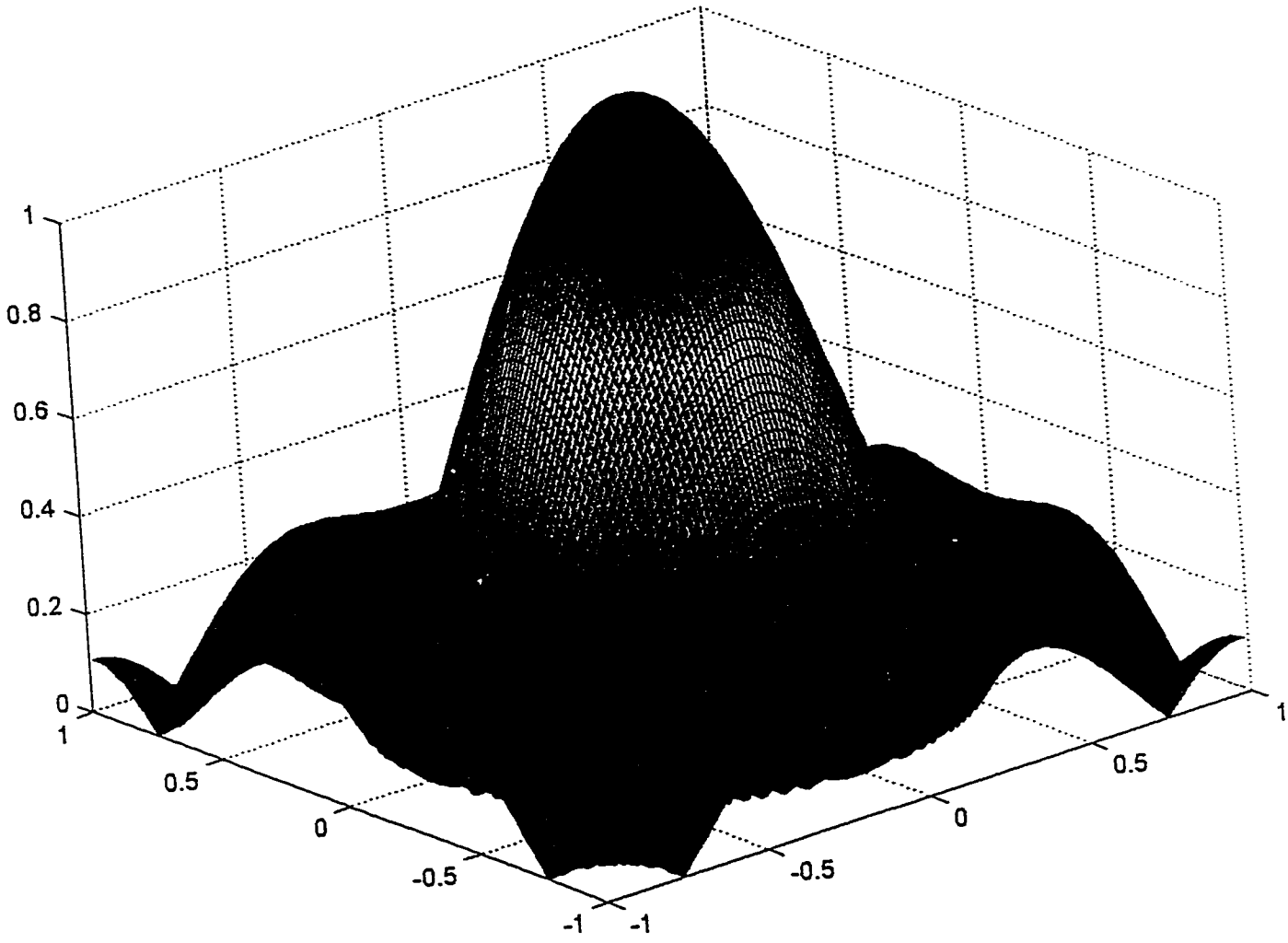

Figure 8-2: Frequency Response of 5x5 FIR Filter

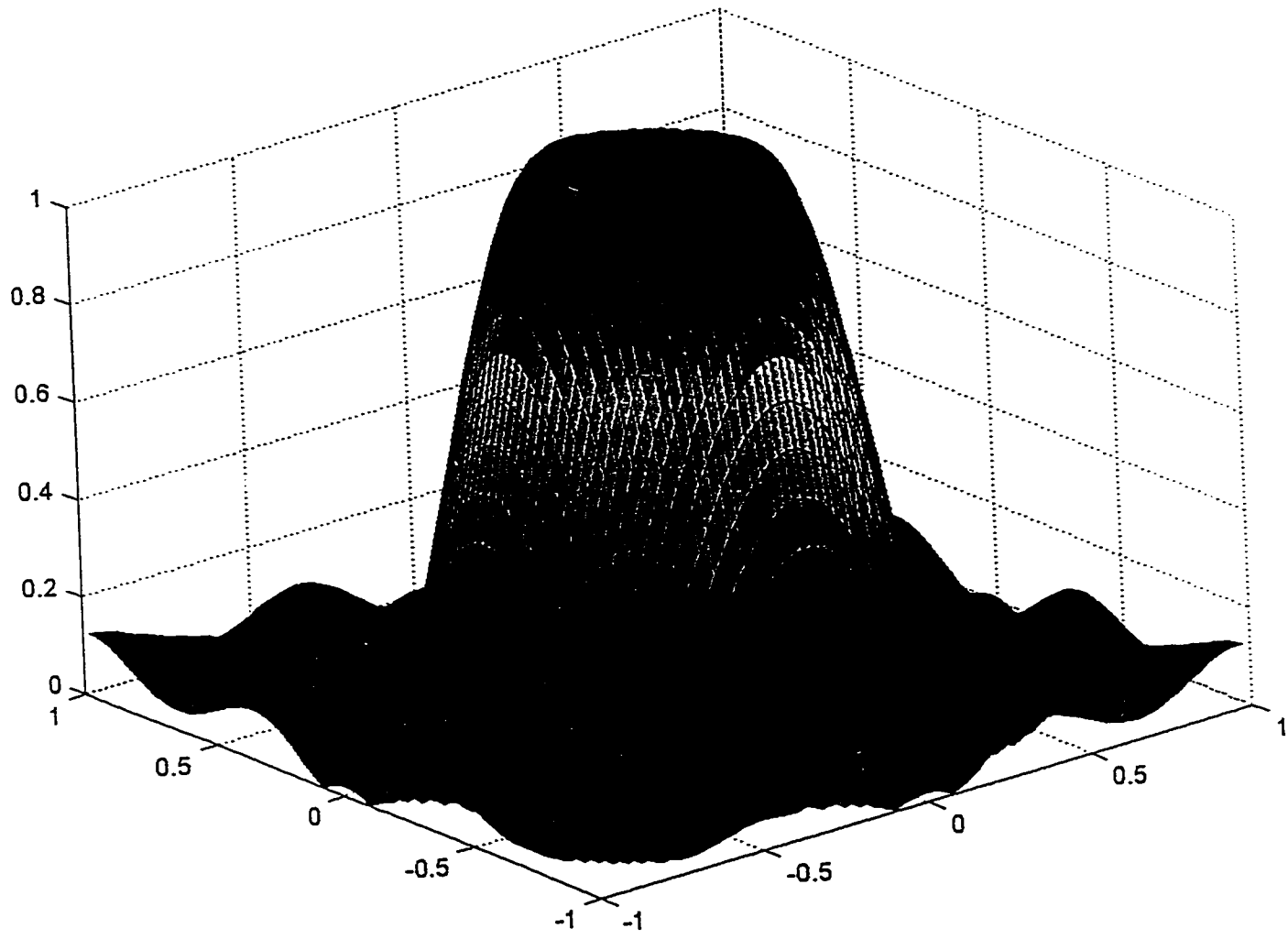

Figure 8-3: Frequency Response of 7x7 FIR Filter

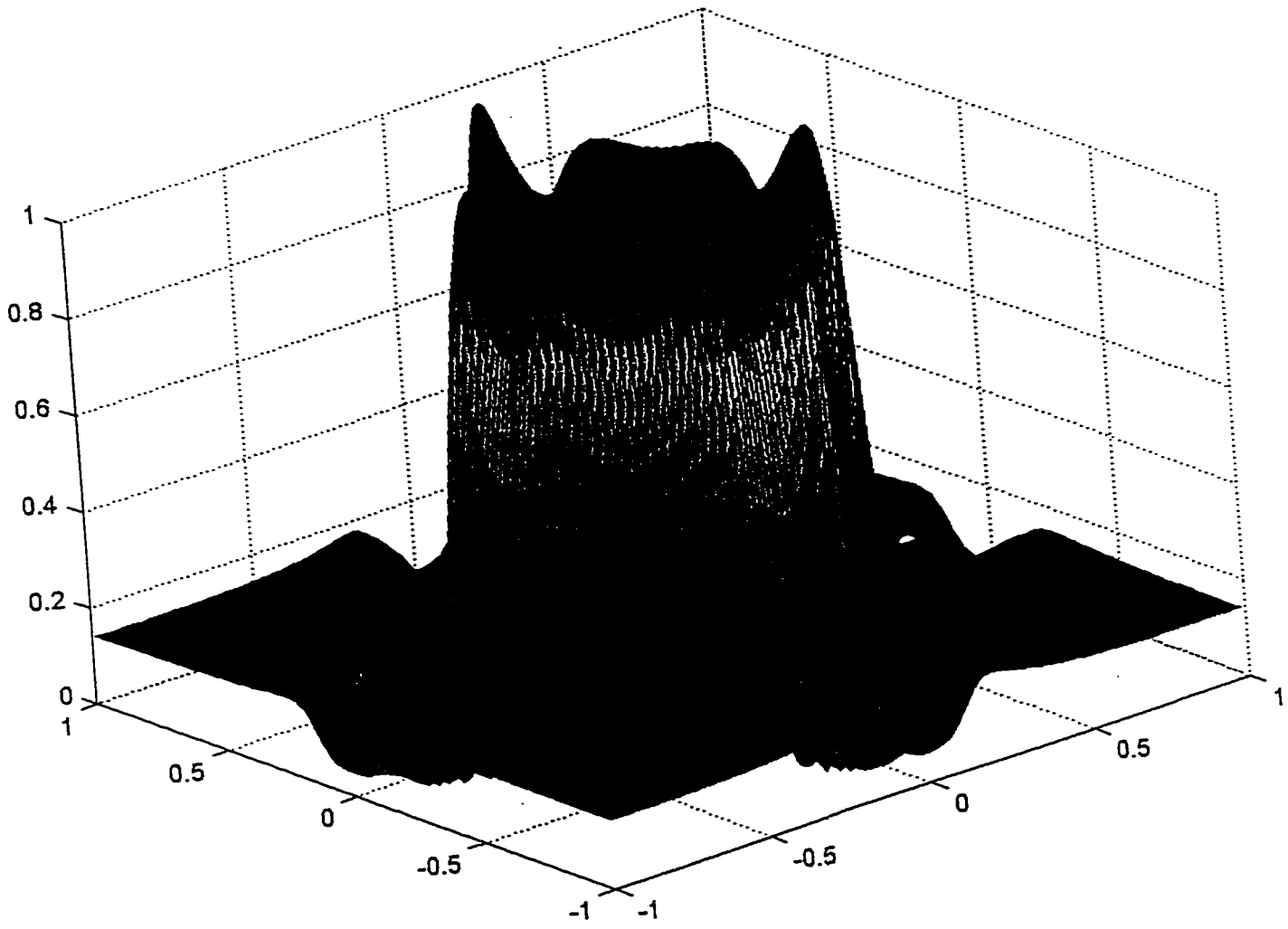

Figure 8-4: Frequency Response of IIR Filter With r=2

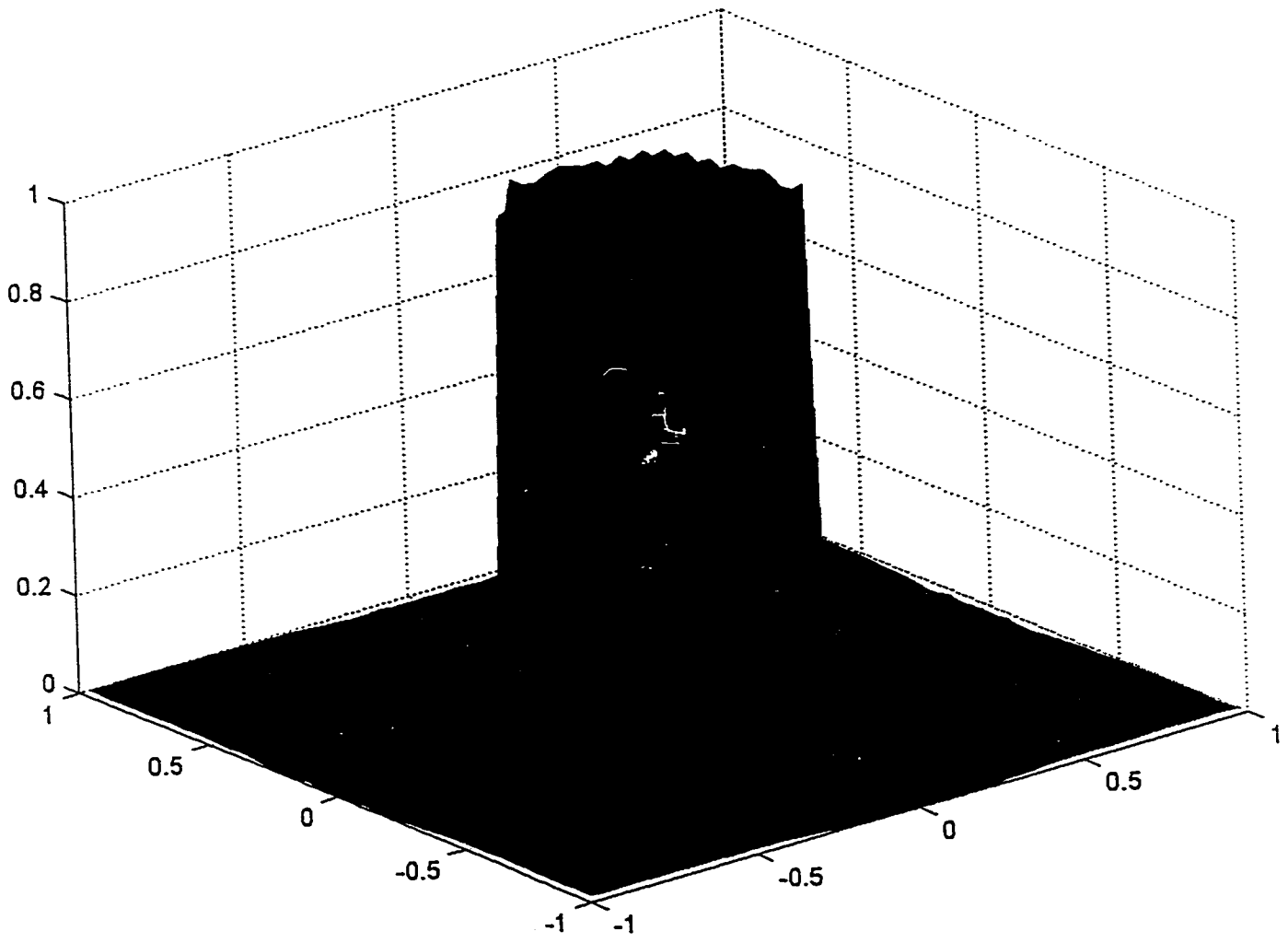

Figure 8-5: Frequency Response of IIR Filter With r=3

Chapter 9

CONCLUSIONS AND RECOMMENDATION FOR FUTURE RESEARCH

9.1 Conclusions

The H_∞-norm approximation has many applications in the field of control and signal processing. In this thesis, iterative algorithms to solve the weighted and the unweighted H_∞-norm approximation problems were proposed. Time domain algorithms based on least squares and output error identification methods have been proposed. In most cases the least squares algorithm is faster and leads to better approximations. Smoothing has been proposed to improve the stability of the algorithms. It improves the computational properties but it is computationally more expensive.

The FFT technique was used to reduce the computation time of the algorithm. Further reduction in computation time is obtained by the frequency domain algorithms. In addition to reduction in computation time, the frequency domain algorithms approximate the systems based on samples of their frequency response. There is no restriction on the frequencies so that it can be used for uniform samples as well as nonuniform samples. They can be used for discrete as well as continuous-time problems. Modified algorithms for unstable and constrained approximation were proposed.

Several examples were used to demonstrate the use of the algorithms and the results were compared to existing methods in the available literature. The proposed algorithms used more computation time but produce better approximations. The quality of the resulted approximation depends on the number of samples N and the maximum number of iterations $l_{\max}$. In general increasing N and $l_{\max}$ lead to better approximation. In general the approximation error may not be monotonically decreasing with the number of iteration. However, as the value of N increases, the performance tends to be better and monotonic. In general most of the error reduction is obtained in the first few iterations. Experience indicates 10 to 50 iterations is sufficient in most cases and only marginal improvements are obtained in the succeeding iterations.

The proposed algorithms and their extensions were used in many applications such as filter design, controller reduction, identification in H_∞ and simultaneous approximation. To design IIR filters to satisfy certain set of specifications, high order FIR models are designed and then the developed algorithms are used to approximate them to obtain lower order IIR filters. Frequency domain algorithms can be used to design filters from the specifications directly. The proposed algorithms allow the design of standard filters as well as general shape filters. The algorithms have been used to design IIR filters with nearly linear phase. An algorithm that

concentrates on reducing the magnitude of the error was developed. This can be used in cases where the restrictions on phase can be relaxed so that small errors in the magnitude can be obtained.

The controller reduction problems that preserve stability with reduced order controllers can be formulated as a weighted H_∞ norm approximation problem. Controller reduction for uncertain systems was formulated as a weighted approximation problem. It was observed that if one is allowed to relax the performance level then better results are obtained if relaxation is done in the design of the controller. Optimal controllers are more difficult to approximate.

The H_∞ norm approximation algorithms can be used as a major part of a new algorithm for identification in H_∞. The frequency domain algorithms can be used to obtain a nominal model that best fits the given frequency response data and an upper bound on the worst case error can be obtained using Theorem 2. Unlike most existing methods where the nominal models are described by their impulse response models, the new algorithm gives rational nominal models with the specified order. Simulation examples showed that the nominal models obtained can be very close to the true models.

The simultaneous approximation problem is well known in mathematical literature. Several problems in identification and controller reduction were formulated as simultaneous approximation problems. Algorithms to solve the SAP based on the frequency domain H_∞ norm approximation algorithms were proposed. Several case studies were solved using the developed algorithms.

The H_∞ norm approximation algorithms have been extended to 2-D and were used to solve 2-D filter design problems.

9.2 Recommendation for Future Research

The availability of efficient algorithms to solve the H_∞ approximation problem make it possible to solve many problems in control and signal processing. In this thesis several application examples were solved using the proposed algorithms and other problems can be solved.

1. In Section 6.4, an idea for general H_∞-norm identification algorithm was proposed. The properties of this algorithm have not been fully investigated.

2. In Chapter 6, frequency domain identification algorithms were proposed. Time domain H_∞-norm identification algorithms have been tried. Initial studies show that the empirical transfer function obtained is noisy even with noise free data.

3. The way in which the error bounds in Section 6.3 may be extended to multivariable systems and combined with Algorithm 3.11 and the H_∞-norm identification algorithm can be extended to MIMO case.

4. In Chapter 7, the simultaneous approximation problem was considered for the SISO case. Extension to the MIMO case is of great interest.

5. In Chapter 3, the proposed H_∞-norm approximation algorithms are solved by solving weighted 2-norm approximation problems and the weights are updated by multiplying the previous weight by the error in that iteration. One possible alternative is to modify the weights as a weighted sum of the previous weight and the error in that iteration. Initial investigation suggests that this approach is comparable in performance to the proposed algorithms. Further investigation is needed.

Bibliography

[1] Al-Saggaf U. M. and Bettayeb M., *Techniques in Optimized Model Reduction For High Dimensional Systems,* Control and Dynamic Systems, Vol. 55, pp. 51-109, 1993.

[2] Genesio R. and Milanese M., *A Note on The Derivation and Use of Reduced Order Models,* IEEE Trans. Auto. Control, Vol. AC-21, pp. 118-122, 1976.

[3] Bosley, M. J. and Lees F. P., *A Survey of Simple Transfer Function Derivations from High Order State-Variable Models,* Automatica, Vol. 8, pp. 765-775, 1972.

[4] Davison E. J., *A Method for Simplifying Linear Dynamic Systems,* IEEE Trans. Auto. Control, Vol. AC-12, pp. 119-121, 1966.

[5] Moore B. C., *Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction,* IEEE Trans. Auto. Control, Vol. AC-26, pp. 17-31, 1981.

[6] Wilson D. A., *Model Reduction for Multivariable Systems,* Int. J. Control, Vol. 20, pp. 57-64, 1974.

[7] El-Attar R. A. and M. Vidyasagar, *Order Reduction by l_1 and l_∞-Norm Minimization,* IEEE Trans. Auto. Control, Vol. AC-23, pp. 731-734, 1978.

[8] Reddy A. S., *A Method for Frequency Domain Simplification of Transfer Functions,* Int. J. Control, Vol. 23, pp. 403-408, 1976.

[9] Luss R., *Optimization in Model Reduction,* Int. J. Control, Vol. 32, 741-, 1980.

[10] Al-Saggaf U. M., *Approximation of Discrete Linear Systems Based on Dual Criterion,* Int. J. System Science, pp. , 1991.

[11] Glover K., *All Optimal Hankel-Norm Approximations of Linear Multivariable Systems and Their L_∞- Error Bounds,* Int. J. Control, Vol. 39, pp. 1115-1193, 1984.

[12] Walsh J. L., *Interpolation and Approximation by Rational Functions in The Complex Domain,* 5th Ed., American Mathematical Society, 1969.

[13] Gutknecht M. H., *On Complex Rational Approximation Part I: The Characterization Problem,* Computational Aspects of Complex Analysis, pp. 79-101, Reidel Publishing Co., 1983.

[14] Thiran J. P. and Istace M. P., *Optimality and Uniqueness Conditions in Complex Rational Chebyshev Approximation with Examples*, Constrained Approximation, pp. 83-103, 1993.

[15] Williams J., *Characterization and Computation of Rational Chebyshev Approximation in The Complex Plane,* SIAM J. Numer. Anal., Vol. 16, No. 5, pp. 819-827, 1979.

[16] Kavranoğlu D. and Al-Amer S., *A New Frequency Domain Computational Scheme for H_∞-Norm Approximation,* CDC-96, pp. 4574-4579, Kobe 1996.

[17] Kavranoğlu D., Al-Amer S. and Bettayeb M., *New Identification Based Weighted H_∞-Norm Approximation Scheme and its Applications to Controller Reduction,* IEE proc. Control Theory Appl. Vol. 144, No. 1, pp. 61-66, 1997.

[18] Kavranoğlu D. and Al-Amer S., *A New Efficient Frequency Domain Algorithm for H_∞ Approximation With Applications to Controller Reduction,* To appear in IEE proc. Control Theory Appl. 1998.

[19] Al-Amer, S. and Kavranoğlu D., *A New Algorithm for Identification in H_∞,* submitted to ACC-99, 1998.

[20] Kailath T., *Linear Systems,* Prentice-Hall, 1990.

[21] Boyd, S. and El Ghaoui, *Linear Matrix Inequalities in System and Control Theory,* SIAM 1994.

[22] Gahinet P., *A Linear Matrix Inequality Approach to H_∞ Control,* Int. J. of Robust and Nonlinear Control, Vol. 4, pp. 421-448, 1994.

[23] Gutknecht, M. H. and Trefethen L. N., *Nonuniqueness of Best Rational Chebyshev Approximation on the Unit Disk*, Journal of Approximation, pp. 257-288, 1983.

[24] Trefethen L. N., *Near Circularity of the Error Curve in Complex Chebyshev Approximation*, Journal of Approximation, pp. 344-367, 1981.

[25] Kavranoğlu D. and Bettayeb M., *Characterization of the Solution to the Optimal L_∞ Model Reduction Problem,* Systems & Control Letters, Vol. 20, pp. 99-107, 1993.

[26] Kavranoğlu D., *Computation of the Solution for the Optimal H_∞ Model Reduction Problem,* Proc. ACC 93, pp. 2190-2194, San Francisco, 1993. Also in IEEE Tran. Auto. Control, Vol. 39, No. 7, pp. 1447-1451, 1994.

[27] Kavranoğlu D. and Bettayeb M., *Characterization and Computation of the Solution to the Optimal L_∞ Approximation Problem,* IEEE Trans. Auto. Control, Vol. AC-39 pp. 1899-1904, 1994.

[28] Kavranoğlu D. and Bettayeb M., *Discrete Time H_∞ Model Reduction Problem,* Proc. 1993 IFAC World Congress, Vol. III, pp. 365-368, Sydney, Australia, 1993.

[29] Kavranoğlu D. and Bettayeb M., *LMI Based Computational Schemes for H_∞ Model Reduction ,* 13th IFAC World Congress, San Francisco, USA, June, 1996.

[30] Mullis C. T. and Roberts R. A., *Synthesis of Minimum Roundoff Noise Fixed Point Digital Filters,* IEEE Trans. circuits and Systems, Vol. 23, pp. 551-562, 1976.

[31] Pernebo L. and Silverman L. M., *Model Reduction via Balanced State Space Representations,* IEEE Trans. Auto. Control, Vol. AC-34, pp. 382-387, 1982.

[32] Enns D., *Model Reduction for Control System Design,* Ph.D. Thesis, Aeronautics and Astronautics Dept., Stanford University, 1984.

[33] Laub A. J., M. T. Heath, C. C. Page and R. C. Ward, *Computation of Balancing Transformation and Other Applications of Simultaneous Diagonalization Algorithm,* IEEE Trans. Auto. Control, Vol. AC-32, pp. 115-122, 1987.

[34] Safonov M. G. and R. Y. Chiang, *A Shur Method for Balanced Truncation,* IEEE Trans. Auto. Control, Vol. AC-34, pp. 729-733, 1989.

[35] Al-Saggaf U. M. and Franklin G. F. , *An Error Bound for Discrete Reduced Order Model of Linear Multivariable Systems,* IEEE Trans. Auto. Control, Vol. AC-32, pp. 815-815, 1987.

[36] Al-Saggaf U. M. and Franklin G. F., *Model Reduction via Balanced Realization: An Extension and Frequency Weighting Techniques,* IEEE Trans. Auto. Control, Vol. AC-33, No. 7, pp. 687-692, 1988.

[37] Hung Y. S. and Glover K., *Optimal Hankel-Norm Approximation of Stable Systems with First Order Stable Weighting Functions,* Systems & Control Letters, Vol. 7, pp. 165-172, 1986.

[38] Zhou K., *Frequency-Weighted H_∞ Norm and Optimal Hankel Norm Model Reduction,* IEEE Trans. Auto. Control, AC-40, pp. 1687-1699, 1995.

[39] Sreeram V., B. D. O. Anderson and A. G. Madievski, New Results on Frequency Weighted Balanced Reduction Technique, Proceedings ACC-96, pp. 4004-4009, Seattle, 1996.

[40] Silverman L. M. and Bettayeb M., *Optimal Approximation of Linear Systems,* Proc. Joint Automatic Control Conference, Paper FA8.A., 1980.

[41] Bettayeb M., L. M. Silverman and M. G. Safonov, *Optimal Approximation of Continuous-Time Systems,* Proc. 19th IEEE Conference Decision Control, pp. 195-198, 1980.

[42] Kung S. Y. and Lin D. W., *Optimal Hankel Norm Model Reduction: Multivariable Systems,* IEEE Trans. Auto. Control, Vol. AC-26, pp. 832-852, 1981.

[43] Kung S. Y. and Lin D. W., *A State Space Formulation for Optimal Hankel-Norm Model Approximation,* IEEE Trans. Auto. Control, Vol. AC-26, pp. 942- 946, 1981.

[44] Lin D. W. and Kung S. Y., *Optimal Hankel Norm Approximation of Continuous-Time Linear Systems,* Rational Approximation in Linear Systems Engineering, pp. 407-431, Birkhauer, Boston, 1983.

[45] Latham G. A. and B. D. O. Anderson, *Frequency Weighted Optimal Hankel-norm Approximation of Stable Transfer Function,* Systems & Control Letters, Vol. 5, pp. 229-236, 1986.

[46] Kavranoğlu D., *Zeroth Order H_∞ Norm Approximation of Multivariable Systems,* Numerical Functional Analysis and Optimization, Vol. 14(1&2), pp. 89-101, 1993.

[47] Helmersson A., *Model Reduction Using LMIs,* Proc. CDC-94, pp. 3217-3222, 1994.

[48] Levy E. C., *Complex Curve Fitting,* IRE Trans. Auto. Control, Vol. AC-4, pp. 37-43, 1959.

[49] Sanathan C. K. and J. Koerner., *Transfer Function Synthesis as a Ratio of Two Complex Polynomials,* IEEE Trans. Auto. Control, Vol. AC-8, pp. 56-58, 1963.

[50] Whitefield A. H., *Transfer Function Synthesis Using Frequency Response Data,* Int. J. Control, Vol. 43, pp. 1413-1426, 1986.

[51] Spanos J. T., *Algorithms for l_2 and l_∞ Transfer Function Curve Fitting,* Proc. AIAA Guid. Navig. and Control Conf. New Orleans, pp. 1739-1747, 1991.

[52] Hakvoort R. G. and P. M. J. Van Den Hof, *Frequency Domain Curve Fitting with Maximum Amplitude Criterion and Guaranteed Stability,* Int. J. Control, Vol. 60, pp. 809-825, 1994.

[53] Payne P. A., *An Improved Technique for Transfer Function Synthesis from Frequency Response Data,* IEEE Trans. Auto. Control, Vol. AC-15, pp. 480-483, 1970.

[54] Cheney E. W. and H. L. Loeb, *Generalized Rational Approximation,* SIAM Numer. Vol. 1, No. 3, pp. 11-25, 1963.

[55] Cheney E. W. and T. H. Southard, *A Survey of Methods for Rational Approximation with Particular Reference to a New Method Based on a Formula of Darboux,* SIAM Review, Vol. 5, No. 3, pp. 219-231, 1963.

[56] L. Barrodale, M. J. D. Powell and F. D. K. Roberts, *The Differential Correction Algorithm for Rational Approximation,* SIAM J. Numer. Anal., Vol. 9, pp. 493-504, 1972.

[57] Cortelazzo G. Mian G. A., and Morandini, *Rational Multiple Criterion Approximation and Rational Complex Approximation by Differential Correction Algorithms,* SIAM J. Sci., Vol.16, No. 4, pp. 974-991, 1995.

[58] Cortelazzo G. De Simone C., and Mian G.A., *IIR Transfer Function Design with Double-Objective Differential Correction Algorithm,* IEEE Trans. Circuit and Systems, pp. 791-796, 1991.

[59] Davis P. J., *Interpolation & Approximation,* Dover 1975.

[60] Lawson C. L., *Contributions to The Theory of Linear Least Maximum Approximation,* Ph.D. Thesis, University of California, 1961.

[61] Cline A. K., *Rate of Convergence of Lawson's Algorithm,* Mathematics of Computations, Vol.26, No. 117, pp. 167-176, 1972.

[62] Rice J. R. and K. Usow, *The Lawson Algorithm and Extensions,* Mathematics of Computations, Vol. 22, No. 101, pp. 118-127, 1968.

[63] Kavranoğlu D., *Elementary Solutions for the H_∞-General Distance Problem-Equivalence of H_2 and H_∞ Optimization Problems,* Ph.D. Thesis, Electrical Engineering Dept., California Institute of Technology, 1989.

[64] Ellacott S. and Williams J., *Linear Chebyshev Approximation in the Complex Plane Using Lawson's Algorithm,* Mathematics of Computation, Vol. 30, pp. 35-44, 1976.

[65] S. Ellacott and J. Williams, *Characterization and Computation of Rational Chebyshev Approximation in the Complex Plane,* SIAM J. Numer. Anal., Vol.13, No. 3, pp. 310-323, 1976.

[66] X. Chen and T. W. Parks, *Design of IIR Filters in The Complex Domain,* IEEE Trans. Acoust., Speech, Signal Processing, Vol. ASSP-38, pp. 910-920, 1990.

[67] Lee, J. H., and C. K. Chen, *Recursive Digital Filter Design in The Complex Domain Using an Efficient Method,* IEEE Trans. Acoust., Speech, Signal Processing, Vol. ASSP-38, pp. 2429-2432, 1992.

[68] Tseng C. Y. *An Efficient Implementation of Lawson's Algorithm with Applications to Complex Chebyshev FIR Filter Design,* IEEE Trans. on Circuit & Systems-II, Vol. 42, pp. 245-260, 1995.

[69] Soderstrom T. and Stoica P., *System Identification,* Prentice-Hall, 1989.

[70] Ljung L., *System Identification: Thorey for the User,* Prentice-Hall, 1987.

[71] Kavranoğlu D. and M. S. Ahmed, *H_∞ Norm Approximation By Identification Techniques,* Proc. CDC-95, pp. 796-801, New Orleans, USA, December 1995.

[72] Anderson B. D. O. and Liu Y., *Controller Reduction: Concepts and Approaches,* IEEE Trans. Auto. Control, Vol. AC-34, pp. 802-812, 1989.

[73] Zhou K., Doyle J. C., and Glover K., *Robust and Optimal Control,* Prentice Hall, 1996.

[74] Parker P. J. and Anderson B. O. D., *Unstable Rational Function Approximation,* Int. J. Control, Vol. 46, No. 5, pp. 1783-1801, 1987.

[75] Meyer D. G., *Fractional Balanced Reduction: Model reduction via fractional representation,* IEEE Trans. Auto. Control, Vol. AC-35, pp. 1341-1345, 1990.

[76] Al-Saggaf U. M., *Model Reduction for Discrete Unstable Systems Based on Generalized Normal Representation,* Int. J. Control, Vol. 55, No. 2, pp. 431-443, 1992.

[77] Kavranoğlu D. and Bettayeb M., *Rational L_∞ Norm Approximation of Multivariable Systems,* Proc. 34rd CDC, pp. 790-795, New Orleans, 1995.

[78] Bettayeb M. and Kavranoğlu D., *Performance Evaluation of a New H_∞ Model Reduction Scheme,* Proc. 32nd CDC, pp. 2913-2914, San Antonio, Texas, 1993.

[79] Deczky, A. G. *Synthesis of nonrecursive digital filters using minimum p-error criteria,* IEEE Trans. Audio Electroacoust, Vol. AU-20, pp. 257-263, 1972.

[80] Cortelazzo G. and M. R. Lighter, *Simultaneous Design in Both Magnitude and Delay of IIR and FIR Filter Based on Multiple Criterion Optimization,* IEEE Trans. Acoust., Speech, Signal Processing, Vol. ASSP-32, pp. 949-967, 1984.

[81] Chottera A. T. and G. A. Jullien, *A Linear Programming Approach to Recursive Digital Filter Design with Linear Phase,* IEEE Trans. Circuits System, Vol. CAS-29, pp. 139-149, 1982.

[82] Helms, H. D., *Digital Filters with Equiripple or Minimax Response,* IEEE Trans. Audio Electroacoust, Vol. AU-19, pp. 87-94, 1971.

[83] Dudgeon D. E., *Recursive Filter Design Using Differential Correction,* IEEE Trans. Acoust., Speech, Signal Processing, Vol. ASSP-22, pp. 443-448, 1974.

[84] Kobayashi T. and S. R. Imai, *Design of IIR Digital Filters with Arbitrary Log Magnitude Functions by WLS Techniques,* IEEE Trans. Acoust. Speech, Signal Processing, Vol. ASSP-38, pp. 247-252, 1990.

[85] Lim Y. C., J. H. Lee, C. K. Chen and R. H. Yang, *A Weighted Least Squares Algorithm for Quasi-Equiripple FIR and IIR Digital Filter Design*, IEEE Trans. on Signal Processing, Vol. 40, No 3, pp. 551-558, 1992.

[86] Inukai T., *A Unified Approach to Optimal Recursive Digital Filters*, IEEE Trans. Circuit System, Vol. CAS-29, pp. 643-649, 1980.

[87] Lee J. H. and C. K. Chen, *Recursive Digital Filter Design in the Complex Domain Using an Efficient Method,* Proc. IEEE International Symposium on Circuits and Systems, pp. 2429-2432, San Diego, 1992.

[88] Beliczynski B., Gryka J. and G. D. Cain, *IIR Filter Design via Hankel-Norm Optimal Approximation of FIR prototype: A streamline approach,* Electron. Lett. , pp. 292-293,1994.

[89] Beliczynski B., I. Kale and G. D. Cain, *Approximation of FIR by IIR Digital Filters:An Algorithm Based on Balanced Model Reduction,* IEEE Trans. on Signal Processing, Vol. 40, No. 3, pp. 532-541, 1992.

[90] Holford S., and P. Agathoklis, *The Use of Model Reduction Techniques for Designing IIR Filters with Linear Phase in The Passband,* IEEE Trans. Signal Processing Vol. 44, pp. 2396-2404, 1996.

[91] Bernstein D. S. and Hyland D. C., *The Optimal Projection Equations for Fixed-order Dynamic Compensation,* IEEE Trans. Auto. Control, AC-29, pp. 1034–1037, 1984.

[92] Gangaas D. , K. R. Bruce, J. D. Blight and U. L. Ly, *Applications of modern synthesis to aircraft control: Three case studies,* IEEE Trans. Auto. Control, Vol. AC-31, 995-1104, 1986.

[93] Goddard P. J. and Glover K., *Performance Preserving Frequency Weighted Controller Approximation: A Coprime Factorization Approach,* Proc. CDC 94, pp. 2720-2725, 1994.

[94] Anderson B. D. O. and Liu Y., *Controller Reduction via Stable Factorization and Balancing,* Int. J. Control, Vol. 4, pp. 507-531, 1986.

[95] Zhou K., and J. Chen, *Performance Bounds For Coprime Factor Controller Reductions,* Proc. ACC 95, pp. 4020-4024, 1995.

[96] Gu, D. W., B. W. Choi and Postlethwaite, *Low-Order H_∞ Suboptimal Controller,* 12th IFAC World Cong. Sydney, Australia, pp. 347-350, 1993.

[97] Li, X. P. and B. C. Chang, *A Parameterization Approach to Reduced Order H_∞ Controller,* CDC-93, San Antonio, pp. 2909-2912, 1993.

[98] Lenz K. E., Khargonekar, and Doyle J. C., *Controller Order Reduction with Guaranteed Stability and Performance,* Proc. ACC, pp. 1697-1698, 1988.

[99] Goddard P. J. and Glover K., *Controller Reduction: Weights for Stability and Performance Preservation,* Proc. CDC-93, pp. 2903-2908, 1993.

[100] Yang, X. H. and Packard A., *A Low Order Controller Design Method,* Proc. CDC 95, pp. 3068-3073, 1995.

[101] Kavranoğlu D. , *Controller Reduction for Uncertain Systems,* Proceeding of 35th CDC, pp. 887-892,1996.

[102] Helmicki A. J., C. A. Jacobson and C. N. Nett. *Identification in H_∞: A Robust Convergent Nonlinear Algorithm.* Proc. international Symposium MTNS-89, pp. 449-463, 1989.

[103] Helmicki A. J., C. A. Jacobson and C. N. Nett, *Least Square Methods for H_∞ Control Oriented System Identification,* Proc. CDC-92, pp. 258-264, 1992.

[104] Helmicki A. J., C. A. Jacobson and C. N. Nett. *Worst Case System Identification in H_∞: Validation of Apriori Information, Essential Optimal Algorithm and Error Bounds.* Proc. CDC-92, pp. 251-257, 1992.

[105] Gu G. and P. P. Khargonkar, *Linear and Nonlinear Algorithms for Identification in H_∞ with Error Bounds,* IEEE Tran. Aut. Control, Vol. AC-37, pp. 952-963, 1992.

[106] Gu G. and Xiong D. and Zhou K., *Identification in H_∞ Using Pick's Interpolation,* Systems & Control Letters, Vol. 20, pp. 263-272, 1993.

[107] Bai, E. W., *Adaptive Quantification of Model Uncertainty By Rational Approximation,* IEEE Trans. Auto. Control, Vol. AC-36, pp. 441-453, 1991.

[108] Akcay H., G. Gu and P. P. Khargnokar, *A Class of Algorithms for Identification in H_∞ : Continuous-Time Case,* IEEE Trans. Auto. Control, Vol. AC-38, pp. 289-294, 1993.

[109] Helmicki A. J., C. A. Jacobson and C. N. Nett, *Worst Case Deterministic Identification in H_∞: The continuous Time Case.* IEEE Trans. Auto. Control, Vol. AC-37 pp. 449-463, 1992.

[110] Partington J. R. *Robust Identification and Interpolation in H_∞,* Int. J. of Control, Vol. 54, No. 5, pp.1281-1290, 1991.

[111] Chen J. and S. Wang, *New Time Domain Algorithms For H_∞ Identification,* CDC-95, pp. 1976-1980, 1995.

[112] Theodor Y. U. Shaked and N. Berman *Time Domain H_∞ Identification,* IEEE Trans. Auto. Control, Vol. AC-41, pp. 1019-1024, 1996.

[113] Zhou, T. and H. Kimura, Time Domain Identification for Robust Control, Systems & Control Letters, Vol. 20, pp. 167-178,1993.

[114] Ahlfors, L. V., Complex Analysis, Third Edition, McGraw-Hill, 1979.

[115] Scheid, R. E. and D. S. Bayard, *A Globally Optimal Minimax Solution for Spectral Overbounding and Factorization,* IEEE Trans. on Auto. Control, Vol. AC-40, pp. 712-716, 1995.

[116] Diaz J. B. and McLaughlin H. W., *Simultaneous Chebyshev Approximation of a Set of Bounded Complex Valued Functions*, Journal of Approximation, pp. 419-432, 1969.

[117] Freilech J. H., *Simultaneous Approximation of a Set of Complex-Valued Function in a Normed Vector Space,* Journal of Approximation, Vol. 29, pp. 179-199, 1980.

[118] Tanimoto S., *A Characterization of Best Simultaneous Approximations,* Journal of Approximation, pp. 359-361, 1989.

[119] Toran P., *Simultaneous Approximation of a Set of Complex Valued Functions in a Normed Vector Space*, Journal of Approximation, pp. 23-85, 1980.

[120] Holland A. S. B. and B. N. Shaney, *Some Remarks on Best Simultaneous Approximation,* Theory of Approximation with Applications, edited by A. G. Law and B. N. Sahney , pp. 661-687, 1987.

[121] Kavranoğlu D., Bettayeb M. and Anjum, M. F., *L_∞-Norm Simultaneous System Approximation,* Int. J. Robust and Nonlinear Control, pp. 999-1014, 1996.

[122] Zhou K., Aravena J. L., Gu G. and Xiong D., *2-D Model Reduction by Quasi-Balanced Truncation and Singular Perturbation,* IEEE Trans. Circuit and Systems II, Vol. 41, pp. 593-602, 1994.

[123] Xiao C. and P. Agathoklis, *Design of 2-D Linear Phase IIR Digital Filters Using 2-D Impulse Response Gramians and Implementation with Low Roundoff Noise and No Overflow Oscillations,* IEEE proc. pp. 425-428, 1996.

[124] Xiao C., *2-D Weighted Impulse Response Gramians and Model Reduction of 2-D Separable Denominator Digital Systems,* pp. 345-350, 1995.

[125] Park M. S. and W. J. Song, *A New Design of 2-D Linear Phase FIR Filters with Finite-Precision Coefficients,* IEEE Trans. Circuit and Systems II, Vol. 41, pp. 478-486, 1994.

[126] Kavranoğlu D. and Bettayeb M., *Weighted L_∞ Norm Approximation with Given Poles Using LMI Techniques and Applications,* Proc. of European Control Conference, Rome, Italy, Volume 1, pp. 567-571, September 1995.

[127] Kavranoğlu D., *H_∞ Norm Approximation of Systems by Constant Matrices and Related Results,* Proc. CDC 92, pp. 3271-3275, Tucson, Arizona, 1993. Also in IEEE Trans. Auto. Control, Vol. 39, pp. 1006-1009, 1994.

[128] Kavranoğlu D. and Bettayeb M., *An Iterative Scheme for Rational L_∞ Approximation,* Proc. of ECCTD-95, Istanbul, Turkey, pp905-908, 1995.

[129] Kavranoğlu D., Bettayeb M. and Anjum, M. F., *Rational L_∞-Norm Approximation of Multivariable Systems,* Proc. of the 34th IEEE CDC, New Orleans, Louisiana, pp. 790-795, 1995.

[130] Bettayeb M. , *Approximation of Linear Systems: New Approaches Based on Singular Value Decomposition,* PhD Thesis, Electrical Engineering Dept., University of Southern California, 1981.

[131] μ-Analysis and Synthesis Toolbox of Matlab.

[132] Wang W., Doyle J., Beck C., and Glover K., *Model Reduction of LFT Systems,* Proc. CDC 91, pp. 1233-1238, 1991.

[133] Wahlberg B., *On the Identification and Approximation of Linear Systems,* PhD thesis, Linkoping, 1987.

[134] Barreto J. A. and Burrus C. S. *Iterative Reweighted Least Squares and the Design of Two-Dimensional FIR Digital Filters,* Proc. IEEE pp. 775-779, 1994.

[135] Zilouchian A. and Carroll R. I., *Model Reduction of 2-D Systems: A State-Space Approach,* Proc. CDC 87, pp. 118-121, 1987.

[136] Chui C. K., *Approximation and Expansions,* Encyclopedia of Physical Science and Technology, pp. 661-687, 1987.

[137] Chen J. and Nett C. N., *The Caratheodory-Fejer Problem and H_∞ Identification, Essentially Optimal Algorithms and Error Bounds,* CDC 94, pp. 4158-4163, 1994.

[138] Dunham C. B., *Simultaneous Chebyshev Approximation of Functions on an Interval,* Proc. Amer. Math . Soc. Vol.18, pp. 472-477, 1967.

[139] Amir D. and Ziegler Z., *Relative Chebyshev Centers in Normed Linear Spaces*, Journal of Approximation, pp. 235-252, 1980.

[140] Weinstein S. E., *Approximations of Functions of Several Variables: Product Chebyshev Approximations*, I. Journal of Approximation, pp. 433-447, 1969.

[141] Wang W. Glover K. and Doyle J., *Stability and Model Reduction of Linear Fractional Transformation Systems*,

[142] Alkhairy A. *On IIR Filter Design*, IEEE Int. Symp. on Circuits & Systems pp.862-864, 1995.

[143] Alkhairy A. *A Complex Chebyshev Approximation Algorithm for FIR Filter Design* , IEEE Int. Conf.on Acoustics, Speech and Signal Processing, pp. 549-551 1994.

[144] Alkhairy A. *An Efficient Method for IIR Filter Design*, IEEE Int. Conf.on Acoustics, Speech and Signal Processing, pp. 569-571, 1994.

[145] Beck C. L., J. Doyle and K. Glover, *Model Reduction of Multidimensional and Uncertain Systems*,IEEE Trans. on Auto. Control, pp. 1466-1477, 1996.

[146] Burnside D. and T. W. Park, *Optimal Minimax Two-Dimensional FIR Design Using a Multiple Simplex Exchange,* Proc. IEEE pp. 593-596, 199x.

[147] Matsuyama K., Okuda M. and Ikehara M., *Approximation of Group Delay Response Using Weighted Least Square Method,* Proceedings IEEE pp. 197-200, 1996.

[148] Bose N. K. and Baus S. , *Two -Dimensional Matrix Pade Approximation: Existence, Nonuniqueness and Recursive Computation* , IEEE Trans. Auto. Control, Vol. AC-25, pp509-514, 1980.

[149] Rice J. R., *The Approximation of Functions* Vol. III: Nonlinear and Multivariate Theory, Addison-Wesley, Readings, Massachusetts, 1969.

[150] Wang D. and A. Zilouchian, *Model Reduction of Discrete Linear Systems via Frequency Domain Balanced Structure,* Proc. ACC-98, pp. 162-166, Philadelphia, 1998.

[151] Anderson B. D. O., *Weighted Hankel-Norm Approximation : Calculation of Bounds.,* Systems & Control Letters, Vol. 7, pp. 247-255, 1986.

[152] Zhou K., *Error Bounds For Frequency Weighted Balanced Truncation and Relative Error Model Reduction,* Proc. CDC-93, pp. 3347-3352, 1993.

[153] Gutknecht M. H., *On Complex Rational Approximation Part II: The Caratheodory-Fejer Method,* Computational Aspects of Complex Analysis, pp. 103-132, Reidel Publishing Co., 1983.

[154] Trefethen L. N., *Chebyshev Approximation on the Unit Disk,* Computational Aspects of Complex Analysis, pp. 309-323, Reidel Publishing Co., 1983.

[155] Bistritz Y. and Langholz G., *A Model Reduction by Chebyshev Rational Approximation in The Complex Plane,* Int. J. Control, Vol. 30, No. 2, pp. 277-2, 1979.

[156] Hayashi E., Trefethen L. N., and Gutknecht M. H., *The CF Table,* Constr. Approx., Vol. 6, pp. 195-223, 1990.

[157] Lim J. S., *Two Dimensional Signals and Image Processing,* Prentice Hall 1990.

[158] Mikhael W. and Yu, H. , *A Linear Approach for Two Dimensional, Frequency Domain, Least Squares, Signal and System Modeling,* IEEE Trans. on Circuit and Systems II, Vol 41, pp. 789-795, 1994.

[159] Lu W. S., Lee E. B. and Zhang Q. T., *Balanced Approximation of Two Dimensional and Delay Differential Systems,* Int. J. Control, Vol. 46, pp. 2199-2218, 1987.

[160] T. Kobayashi and S. R. Imai, *Design of ARMA digital filters with arbitrary log magnitude functions by WLS techniques,* Proc. of , pp. 1447-1450, 1990.